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M^r Nouredine CHALLALI

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Processus pseudo-presque automorphes et équations différentielles stochastiques

Devant le jury d'examen composé de :

Mr. Mohamed Morsli ;	Professeur ;	U.M.M.T.O ;	Président
Mme. Fazia Bedouhene ;	Professeur ;	U.M.M.T.O ;	Rapporteur
Mr. Paul Raynaud De Fitte ;	Professeur ;	Univ. Rouen ;	Rapporteur
Mr. Brahim Mezerdi ;	Professeur ;	Univ. Biskra ;	Examinateur
Mr. Denis Pennequin ;	Professeur ;	Univ. Paris 1 ;	Examinateur
Mr. Jean-Baptiste Bardet ;	M. Conférences HDR ;	Univ. Rouen ;	Examinateur

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*I dedicate this doctoral
dissertation to all who
are dear to me.*

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General introduction

Historical preamble

The almost periodicity is an old notion whose traces date to ancient Greece, starting from Hipparchus theories up to the celestial model of Ptolemy. This last model, taking into account the observations of that time, was intended to describe the periodic cycles of the celestial bodies of our solar system by decomposing them into elementary cycles and thanks to this idea which essentially characterizes the almost periodic functions by using trigonometric polynomials to approach them. But it took time until the works of H. Bohr, published between 1923 and 1926, for a formal and detailed study of the class of almost periodic functions. Since then, numerous generalizations have been made of this last class of functions, as varied as the distances or topologies used, among which we can cite the classes of Weyl, Stepanov and Besicovitch of almost periodic functions [11, 19, 43, 2, 26] and the class of almost automorphic functions [20, 21]. Later, pseudo almost periodic functions were invented by Zhang [85, 86, 87] and the generalization of this concept to pseudo almost automorphic functions was investigated in [55] which opened the field to other generalizations of almost periodic functions class, thus classes of weighted pseudo almost periodic functions and weyl-like, Stepanov-like and Besicovich- like pseudo periodic functions were defined with their analogue in almost automorphy [17, 34, 25, 38, 16, 37, 35, 83]. The question of studying the concept of almost periodic or almost automorphic stochastic processes arises naturally in connection with stochastic differential equations. As can be seen in Tudor's survey [77], almost periodicity forks into many different notions when applied to stochastic processes : almost periodicity in probability, in p-mean, in one-dimensional distributions, in finite dimensional distributions, in distribution, almost periodicity of moments, etc. These notions are not all comparable : for example, almost periodicity in distribution does not imply almost periodicity in probability, and the converse implication is false too [10]. The situation for almost automorphy is no different. Furthermore, things become even more complicated if one takes into account different generalizations of almost automorphy (which have their analogue in almost periodicity) : on the one hand, changing the mode of convergence in the definition leads to the notions of Stepanov-like, Weyl-like and Besicovitch-like almost automorphy. On the other hand, for the study of asymptotic properties of functions, it is natural to consider functions which are the sum of an almost automorphic function and of a function vanishing in some sense at infinity. In this way, one gets the notions of asymptotically almost automorphic functions and of pseudo almost automorphic functions and their weighted variants. Each of these notions can be interpreted in different manners for stochastic processes, exactly in the same way as for "plain" almost periodicity and almost automorphy. It is an objective of this work to clarify the hierarchy of those various concepts. We did not try to list all possible variants, one can imagine many other extensions and combinations, as the reader will probably do. This is rather a preliminary groundwork, in which we investigate some notions we think particularly useful. For example, when we describe the different modes of pseudo almost automorphy in distribution, we concentrate on a stronger notion, which is not purely "distributional" but seems to be the relevant one for stochastic differential equations. As we have in view applications in spaces of probability measures, which do not have a vector space structure and whose topology can be described by different non uniformly equivalent metrics, A natural application of these concepts is the study of stochastic differential equations with almost automorphic or almost periodic coefficients. We provide examples of stochastic semilinear evolution equations, with almost periodic and with almost periodic coefficients, whose unique bounded solution is almost automorphic (respectively almost periodic) in distribution. It is another objective of this work to point out a common error in many papers which claim the existence of nontrivial solutions which are almost automorphic or almost periodic in quadratic mean. We show by a counterexample borrowed from [60] that this claim is false, even for several extensions of almost automorphy or almost periodicity. .

Context of a work

The study of almost periodic random functions has a relatively long story, starting from Slutsky [72] in 1938, who focused on conditions for weakly stationary random processes to have almost periodic trajectories in Besicovitch's sense. The investigation of almost periodicity in probability was initiated later by the Romanian school [66,24,68]. It is only in the late eighties that almost periodicity in distribution was considered, again by the Romanian school (mainly by Constantin Tudor), in connection with the study of stochastic differential equations with almost periodic coefficients [47,61,76,7,28]. Starting from 2007, many papers appeared, claiming the existence of square-mean almost periodic solutions to almost periodic semilinear stochastic evolution equations, using a fixed point method. Despite the counterexamples given in [60], new papers in this vein continue to be published. The story of almost automorphy and its generalizations is much shorter. Almost automorphic functions were invented by Bochner since 1955 [20] (the terminology stems from the fact that they were first encountered in the context of differential geometry on real or complex manifolds). Almost automorphic stochastic processes and their generalizations seem to have been investigated only since 2010, starting with [44], which was followed by many other papers. Most of these papers claim almost automorphy (or one of its generalizations) in square mean for solutions to stochastic equations. There are only few papers we are aware of [46, 57] which investigate almost automorphy in a distributional sense. Recently, the notion of almost automorphic random functions in probability has been introduced by Ding, Deng and N'Guérékata

Structure of the dissertation

This manuscript is divided into two parts and each part is composed of two chapters. The first part of the thesis is consecrated, on the one hand, to the introduction of almost automorphy notions and the stepanov almost perioicity for the functions and the stochastic processes and on the other hand to the various useful superposition theorems. The second part of the thesis which is also composed of two chapters, is devoted to document the results obtained by the study of the solutions of a stochastic differential equation with almost automorphic terms for the first chapter and Stepanov almost periodic for the second. The distribution of our work in chapter is given hereafter

Chapter 1 :In Section 1.1, we present the concept of almost automorphy and some of its generalizations : μ-pseudo almost automorphy, Stepanov-like, Weyl-like and Besicovitch-like μ-pseudo almost automorphy. Our setting is that of functions of a real variable with values in a metrizable space. Metrizability seems a sufficiently general frame to investigate almost automorphy in many useful spaces of probability measures, while avoiding complications. We show that almost automorphy and a slightly generalized notion of pseudo almost automorphy can be defined in a topological way, without any reference to a metric nor to a vector structure. In Section 1.2.4, we investigate several notions of almost automorphy and pseudo almost automorphy for stochastic processes. First, we investigate almost automorphy and its variants in pth mean : the stochastic processes are seen as almost automorphic (or, more generally, μ-pseudo almost automorphic) functions from $\mathbb{R}$ to L^p, $p \geq 0$ ($p = 0$ corresponds to almost automorphy and its variants in probability). Then we move to almost automorphy in distribution and its variants. For $p > 0$, we introduce the notion of almost automorphy in p-distribution, which is obtained by adding to the preceding notions a condition of p-uniform integrability. For μ-pseudo almost automorphy, the situation becomes even more complicated, because there are several ways to take into account the ergodic part. We introduce also the notion of processes which are μ-pseudo almost automorphic in p-distribution, which are the sum of a process which is almost automorphic in p-distribution and a process which is μ-ergodic in pth mean. We also carry out a comparison of these notions of (generalized) almost automorphy for stochastic processes : in probability, in pth mean, and in p-distribution. In Section 1.2, we investigate several notions of Stepanov almost periodicity in Lebesgue measure, and Stepanov (μ-pseudo) almost periodicity for metric-valued functions. We see in particular that almost periodicity in Stepanov sense depends on the uniform structure of the state space. Next, we proceed like for section

1.2.4, we present various definitions of stochastic processes of almost periodic type.

Chapter 2 : In the section 2.1, we study the superposition operators (also called Nemytskii operators) between spaces of almost automorphic (compact almost automorphic, and μ-pseudo compact almost automorphic) processes in distribution. In the section 2.2, we study some properties of parametric functions, especially Nemytskii's operators $\mathscr{N}(f)(x) := [t \mapsto f(t,x(t))]$ built on $f : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ in the space of Stepanov (μ-pseudo) almost periodic functions.

Chapter 3 : In this chapter, we consider two semilinear stochastic evolution equations in a Hilbert space. The first one has almost automorphic coefficients, and the second one has μ-pseudo almost automorphic coefficients. We show that each equation has a unique mild solution which is almost automorphic in 2-distribution in the first case, and μ-pseudo almost automorphic in 2-distribution in the second case.

Chapter 4 : The aim of this chapter is to study existence and uniqueness of bounded mild solutions to a semilinear stochastic evolution equation on a Hilbert separable space with Stepanov (μ-pseudo) almost periodic terms, satisfying Lipschitz and growth conditions. In the case of uniqueness, the solution can be (μ-pseudo) almost periodic in 2-distribution. Our approach is inspired from Kamenskii et al. [49], Da Prato and Tudor [28], and [9]. The major difficulty is the treatment of the limits of the terms provided by the Bochner criterion in Stepanov sense. Thanks to an application of Komlós's Theorem [50], this difficulty dissipates by showing that those limits inherit the same properties of the first terms. Finally, Section 4.2 is devoted to some remarks and conclusions about the problem of existence of purely Stepanov almost periodic solutions. We showed by a simple example in a one-dimensional setting, that one can obtain bounded purely Stepanov almost periodic solutions when the forcing term is purely almost periodic in Lebesgue measure.

Première partie

Contribution to superposition theorems for almost periodic and almost automorphic type functions and processes

CHAPITRE 1

Almost periodic type and almost automorphic type functions and processes

In the sequel, $\mathbb{X}$ and $\mathbb{Y}$ are metrizable topological spaces. When no confusion may arise, we denote by $\mathfrak{d}$ a distance on $\mathbb{X}$ (respectively on $\mathbb{Y}$) which generates the topology of $\mathbb{X}$ (respectively $\mathbb{Y}$). Most of our results depend only on the topology of those spaces, not on the choice of particular metrics. When $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, their norms are indistinctly denoted by $\|.\|$, and $\mathfrak{d}$ is assumed to result from $\|.\|$.
We denote by $C(\mathbb{X}, \mathbb{Y})$ the space of continuous functions from $\mathbb{X}$ to $\mathbb{Y}$. When this space is endowed with the topology of uniform convergence on compact subsets of $\mathbb{X}$, it is denoted by $C_k(\mathbb{X}, \mathbb{Y})$.
For a continuous function $f : \mathbb{R} \to \mathbb{X}$, we define its *translation mapping*

$$\widetilde{f} : \begin{cases} \mathbb{R} & \to & C(\mathbb{R}, \mathbb{X}) \\ t & \to & f(t+.). \end{cases}$$

1.1 Almost automorphy type functions and processes

1.1.1 Almost periodicity and almost automorphy

Almost periodicity We say that a continuous function $f : \mathbb{R} \to \mathbb{X}$ is *almost periodic* if, for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least an *ε-almost period*, that is, a number τ for which

$$\mathfrak{d}\left(f(t+\tau), f(t)\right) \le \varepsilon, \text{ for all } t \in \mathbb{R}.$$

We denote by $AP(\mathbb{R}, \mathbb{X})$ the space of $\mathbb{X}$-valued almost periodic functions.
By a result of Bochner [20], $f : \mathbb{R} \to \mathbb{X}$ is almost periodic if, and only if, the set

$$\left\{ \widetilde{f}(t), t \in \mathbb{R} \right\} = \left\{ f(t+.), t \in \mathbb{R} \right\}$$

is totally bounded in the space $C(\mathbb{R},\mathbb{X})$ endowed with the norm $\|.\|_\infty$ of uniform convergence.
Another very useful characterization is *Bochner's double sequence criterion* [20] : f is almost periodic if, and only if, it is continuous and, for every pair of sequences (t'_n) and (s'_n) in $\mathbb{R}$, there are subsequences (t_n) of (t'_n) and (s_n) of (s'_n) respectively, with same indexes, such that, for every $t\in\mathbb{R}$, the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+t_n+s_m) \text{ and } \lim_{n\to\infty} f(t+t_n+s_n), \tag{1.1}$$

exist and are equal. This very useful criterion shows that the set $AP(\mathbb{R},\mathbb{X})$ depends only on the topology of $\mathbb{X}$, i.e. it does not depend on any uniform structure on $\mathbb{X}$, in particular it does not depend on the choice of any norm (if $\mathbb{X}$ is a vector space) or any distance on $\mathbb{X}$.

Almost automorphy Almost automorphic functions were introduced by Bochner [20] and studied in depth by Veech [79], see also the monographs [71, 65, 63] for applications to differential equations. A continuous mapping $f:\mathbb{R}\to\mathbb{X}$ is said to be *almost automorphic* if, for every sequence (t'_n) in $\mathbb{R}$, there exists a subsequence (t_n) such that, for every $t\in\mathbb{R}$, the limit

$$g(t)=\lim_{n\to\infty} f(t+t_n) \tag{1.2}$$

exists and

$$\lim_{n\to\infty} g(t-t_n)=f(t). \tag{1.3}$$

The range R_f of f is then relatively compact, because we can extract from every sequence $(f(t_n))$ in R_f a convergent subsequence.
Clearly, the space of almost automorphic $\mathbb{X}$-valued functions depends only on the topology of $\mathbb{X}$.
Almost automorphic functions generalize almost periodic functions in the sense that f is almost periodic if, and only if, the above limits are uniform with respect to t.
Note that, in (1.2) and (1.3), the limit function g is not necessarily continuous. Let us consider the following property :
(**C**) For any choice of (t'_n) and (t_n), the function g of (1.2) and (1.3) is continuous.
Functions satisfying (**C**) are called continuous almost automorphic functions in [79]. They were re-introduced by Fink [**?**] under the name of *compact almost automorphic* functions. This terminology is now generally adopted, so we stick to it.
It has been shown by Veech in [79, Lemma 4.1.1] (see also [62, Theorem 2.6] and [64, 58]) that, if f satifies (**C**), f is uniformly continuous. The proof of Veech is given in the case when $\mathbb{X}$ is the field of complex numbers, but it extends to any metric space.
Furthermore, f satifies (**C**) if, and only if, the convergence in (1.2) and (1.3) is uniform on the compact intervals. We denote by $AA_c(\mathbb{R},\mathbb{X})$ the subspace of functions satisfying (**C**).
We have the inclusions

$$AP(\mathbb{R},\mathbb{X})\subset AA_c(\mathbb{R},\mathbb{X})\subset AA(\mathbb{R},\mathbb{X}).$$

All these spaces depend only on the topological structure of $\mathbb{X}$ and not on its metric.

Almost automorphic functions depending on a parameter Following [55], we say that a function $f : \mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ is *almost automorphic with respect to the first variable, uniformly with respect to the second variable in bounded subsets of* $\mathbb{Y}$ (respectively *in compact subsets of* $\mathbb{Y}$) if, for every sequence (t'_n) in $\mathbb{R}$, there exists a subsequence (t_n) such that, for every $t\in\mathbb{R}$ and every $y\in\mathbb{Y}$, the limit

$$g(t,y) = \lim_{n\to\infty} f(t+t_n,y)$$

exists and, for every bounded (respectively compact) subset B of $\mathbb{Y}$, the convergence is uniform with respect to $y\in B$, and if the convergence

$$\lim_{n\to\infty} g(t-t_n,y) = f(t,y)$$

holds uniformly with respect to $y\in B$. We denote by $\mathrm{AAU}_b(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\mathrm{AAU}_c(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ respectively the spaces of such functions.
Similarly, one can define the spaces of functions $f : \mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ which are *compact almost automorphic with respect to the first variable, uniformly with respect to the second variable in bounded* (or *in compact*) *subsets of* $\mathbb{Y}$. We denote these spaces by $\mathrm{AA}_c\mathrm{U}_c(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\mathrm{AA}_c\mathrm{U}_b(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ respectively.
These notions are different from the notion of functions almost automorphic uniformly in y defined in [18, 16].

Proposition 1.1.1. *Let $f\in AA_cU_c(\mathbb{R}\times\mathbb{Y},\mathbb{X})$. Assume that f is continuous with respect to the second variable. Then f is continous on $\mathbb{R}\times\mathbb{Y}$, and, for every compact subset K of $\mathbb{Y}$, f is uniformly continuous on $\mathbb{R}\times K$.*

Proof : For simplicity, we use the same notation $\mathfrak{d}$ for distances on $\mathbb{Y}$ and $\mathbb{X}$ which generate the topologies of $\mathbb{Y}$ and $\mathbb{X}$ respectively.
First step Let us show that f is jointly continuous. Let $(t,x)\in\mathbb{R}\times\mathbb{Y}$, and let (t_n,x_n) be a sequence in $\mathbb{R}\times\mathbb{Y}$ which converges to (t,x). Let $\varepsilon>0$. The set $K=\{x_n; n\in\mathbb{N}\}\cup\{x\}$ is compact, thus there exists $N_1\in\mathbb{N}$ such that, for any $y\in K$,

$$n,m\geq N_1 \Rightarrow \mathfrak{d}(f(t_n,y), f(t_m,y)) < \varepsilon/3.$$

Now, there exists $N_2\in\mathbb{N}$ such that

$$n\geq N_2 \Rightarrow \mathfrak{d}(f(t_{N_1},x_n), f(t_{N_1},x)) < \varepsilon/3.$$

We deduce, for $n\geq (N_1\vee N_2)$,

$$\begin{aligned}\mathfrak{d}(f(t_n,x_n), f(t,x)) &\leq \mathfrak{d}(f(t_n,x_n), f(t_{N_1},x_n))\\ &\quad + \mathfrak{d}(f(t_{N_1},x_n), f(t_{N_1},x)) + \mathfrak{d}(f(t_{N_1},x), f(t,x)) < \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon,\end{aligned}$$

which proves the continuity of f.

Second step Let (t'_n) be a sequence in $\mathbb{R}$. Let (t_n) be a subsequence of (t'_n) such that, for every $y\in\mathbb{Y}$, and for every $t\in\mathbb{R}$, the limit

$$g(t,y) = \lim_{n\to\infty} f(t+t_n,y)$$

exists, uniformly with respect to y in compact subsets of $\mathbb{Y}$, and

$$\lim_{n\to\infty} g(t-t_n,y) = f(t,y).$$

By our hypothesis, for each $y \in \mathbb{Y}$, the function $g(.,y)$ is continuous. A similar reasoning to that of the first step shows that g is continuous on $\mathbb{R}\times\mathbb{Y}$. Indeed, let $(t,x)\in\mathbb{R}\times\mathbb{Y}$, and let (s_k,x_k) be a sequence in $\mathbb{R}\times\mathbb{Y}$ such that $(t+s_k,x_k)$ converges to (t,x). Let $\varepsilon>0$. The set $K=\{x_n;\, n\in\mathbb{N}\}\cup\{x\}$ is compact, thus there exists an integer N such that, for every $y\in K$,

$$n\geq N \Rightarrow \mathfrak{d}\Big(g(t,y),\, f(t+t_n,y)\Big) < \varepsilon/3.$$

By continuity of f at the point $(t+t_N,x)$, there exists $N'\in\mathbb{N}$ such that

$$k\geq N' \Rightarrow \mathfrak{d}\Big(f(t+s_k+t_N,x_k),\, f(t+t_N,x)\Big) < \varepsilon/3.$$

We have thus, for $k\geq N'$,

$$\begin{aligned}
\mathfrak{d}\Big(g(t+s_k,x_k),g(t,x)\Big) \leq & \mathfrak{d}\Big(g(t+s_k,x_k),\, f(t+s_k+t_N,x_k)\Big)\\
&+\mathfrak{d}\Big(f(t+s_k+t_N,x_k),\, f(t+t_N,x)\Big)\\
&+\mathfrak{d}\Big(f(t+t_N,x),g(t,x)\Big)\\
<& \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.
\end{aligned}$$

Third step Let K be a compact subset of $\mathbb{Y}$. Assume that f is not uniformly continuous on $\mathbb{R}\times K$. We can find two sequences (s_n,x_n) and (t_n,y_n) in $\mathbb{R}\times K$ such that $(s_n-t_n)+\mathfrak{d}(x_n,y_n)$ converges to 0 and $\mathfrak{d}\big(f(s_n,x_n),\, f(t_n,y_n)\big)>2\delta$ for some $\delta>0$ and for all $n\in\mathbb{N}$. By compactnes of K, and extracting if necessary a subsequence, we can assume that (x_n) and (y_n) converge to a common limit $x\in K$. We have thus

$$\begin{aligned}
\liminf_{n\to\infty}\mathfrak{d}\Big(f(s_n,x_n),\, f(s_n,x)\Big)+\liminf_{n\to\infty}\mathfrak{d}\Big(f(t_n,y_n),\, f(t_n,x)\Big)&\\
\geq \liminf_{n\to\infty}\mathfrak{d}\Big(f(s_n,x_n),\, f(t_n,y_n)\Big) &> 2\delta,
\end{aligned}$$

which implies that at least one term in the left hand side is greater than δ. So, we can assume, without loss of generality, that

$$\liminf_{n\to\infty}\mathfrak{d}\Big(f(t_n,y_n),\, f(t_n,x)\Big) > \delta. \tag{1.4}$$

Extracting if necessary a further subsequence, we can assume also that there exists a function $g:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ such that

$$\lim_{n\to\infty} f(t_n,y) = g(0,y)$$

uniformly with respect to $y\in K$. We have proved in the second step that g is continuous. But, then, we have

$$\limsup_{n\to\infty} \mathfrak{d}\Big(f(t_n,y_n), f(t_n,x)\Big)$$
$$\leq \limsup_{n\to\infty} \Big(\mathfrak{d}\Big(f(t_n,y_n), g(0,y_n)\Big) + \mathfrak{d}\Big(g(0,y_n), g(0,x)\Big) + \mathfrak{d}\Big(g(0,x), f(t_n,x)\Big)\Big) = 0,$$

which contradicts (1.4). ■

1.1.2 (Weighted) pseudo almost automorphy

Pseudo almost periodic functions were invented by Zhang [84,85,86,87]. The generalization of this concept to pseudo almost automorphic functions was investigated in [55]. To define pseudo almost automorphy, we need another class of functions. Assume for the moment that $\mathbb{X}$ is a Banach space. Let

$$\mathscr{E}(\mathbb{R},\mathbb{X}) = \left\{ f \in \mathrm{BC}(\mathbb{R},\mathbb{X});\ \lim_{r\to\infty} \frac{1}{2r} \int_{[-r,r]} \|f(t)\|\, dt = 0 \right\},$$

where $\mathrm{BC}(\mathbb{R},\mathbb{X})$ denotes the space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$. We say that a continuous function $f : \mathbb{R} \to \mathbb{X}$ is *pseudo almost automorphic* if it has the form

$$f = g + \Phi, \quad g \in \mathrm{AA}(\mathbb{R},\mathbb{X}), \quad \Phi \in \mathscr{E}(\mathbb{R},\mathbb{X}). \tag{1.5}$$

The space of $\mathbb{X}$-valued pseudo almost automorphic functions is denoted by $\mathrm{PAA}(\mathbb{R},\mathbb{X})$. Weighted pseudo almost automorphic functions were introduced by Blot et al. in [14] and later generalized in [16]. They generalize the weighted pseudo almost periodic functions introduced by Diagana [34, 35, 37], see also [17]. Let μ be a Borel measure on $\mathbb{R}$ such that

$$\mu(\mathbb{R}) = \infty \text{ and } \mu(I) < \infty \text{ for every bounded interval } I. \tag{1.6}$$

We define the space $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ of *μ-ergodic* $\mathbb{X}$-valued functions by

$$\mathscr{E}(\mathbb{R},\mathbb{X},\mu) = \left\{ f \in \mathrm{BC}(\mathbb{R},\mathbb{X});\ \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\|\, d\mu(t) = 0 \right\}.$$

The space $\mathrm{PAA}(\mathbb{R},\mathbb{X},\mu)$ of *μ-pseudo almost automorphic functions* with values in $\mathbb{X}$ is the space of continuous functions $f : \mathbb{R} \to \mathbb{X}$ of the form

$$f = g + \Phi, \quad g \in \mathrm{AA}(\mathbb{R},\mathbb{X}), \quad \Phi \in \mathscr{E}(\mathbb{R},\mathbb{X},\mu). \tag{1.7}$$

The space $\mathrm{PAA}(\mathbb{R},\mathbb{X},\mu)$ contains the *asymptotically almost automorphic* functions, that is, the functions of the form

$$f = g + \Phi, \quad g \in \mathrm{AA}(\mathbb{R},\mathbb{X}), \quad \lim_{|t|\to\infty} \|\Phi(t)\| = 0,$$

see [16, Corollary 2.16].

Note that, contrarily to (1.5), the decomposition (1.7) is not necessarily unique [16, Remark 4.4 and Theorem 4.7], even in the case of weighted almost periodic functions [54, 17]. A sufficient condition of uniqueness of the decomposition is that $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ be translation invariant. This is the case in particular if Condition (**H**) of [16] is satisfied :

(**H**) For every $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that $\mu(A+\tau)\le\beta\mu(A)$ whenever A is a Borel subset of $\mathbb{R}$ such that $A\cap I=\emptyset$.

The following elementary lemma will prove useful.

Lemma 1.1.2. *Let $f\in\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$, with μ satisfying* (1.6)*. There exists a sequence (t_n) in $\mathbb{R}$ such that $(|t_n|)$ converges to ∞ and $(f(t_n))$ converges to 0. If furthermore*

$$\liminf_{r\to\infty}\frac{\mu([0,r])}{\mu([-r,r])}>0, \tag{1.8}$$

one can choose (t_n) converging to $+\infty$.

Proof : Observe first that, for every bounded interval I, the function f satisfies

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r]\setminus I)}\int_{[-r,r]\setminus I}\|f(t)\|\,d\mu(t)=0$$

(see [16, Theorem 2.14] for a stronger result). Assume that the first part of the lemma is false. There exist $\varepsilon>0$ and $R>0$ such that, for $|t|\ge R$, $\|f(t)\|\ge\varepsilon$. Then we have

$$\begin{aligned}0=\lim_{r\to\infty}\frac{1}{\mu([-r,r]\setminus[-R,R])}&\int_{[-r,r]\setminus[-R,R]}\|f(t)\|\,d\mu(t)\\ &\ge\lim_{r\to\infty}\varepsilon\frac{\mu([-r,r]\setminus[-R,R])}{\mu([-r,r]\setminus[-R,R])}=\varepsilon,\end{aligned}$$

a contradiction.

In the case when (1.8) is satisfied, we have

$$\begin{aligned}&\limsup_{r\to\infty}\frac{1}{\mu([0,r]\setminus[0,R])}\int_{[0,r]\setminus[0,R]}\|f(t)\|\,d\mu(t)\\ &\quad\le\limsup_{r\to\infty}\frac{\mu([-r,r]\setminus[-R,R])}{\mu([0,r]\setminus[0,R])}\frac{1}{\mu([-r,r]\setminus[-R,R])}\int_{[-r,r]\setminus[-R,R]}\|f(t)\|\,d\mu(t)\\ &\quad=\limsup_{r\to\infty}\frac{\mu([-r,r])}{\mu([0,r])}\frac{1}{\mu([-r,r]\setminus[-R,R])}\int_{[-r,r]\setminus[-R,R]}\|f(t)\|\,d\mu(t)\le 0.\end{aligned}$$

Then we only need to reproduce the reasoning of the first part of the lemma, replacing $[-r,r]$ by $[0,r]$. ∎

Pseudo almost automorphic functions depending on a parameter Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6). We say that a continuous function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ is *μ-ergodic with respect to the first variable, uniformly with respect to the second variable in bounded subsets of* $\mathbb{Y}$ (respectively *in compact subsets of* $\mathbb{Y}$) if, for every $x\in\mathbb{Y}$, $f(.,x)$ is μ-ergodic, and the convergence of $1/\mu([-r,r])\int_{-r}^{r}\|f(t,x)\|\,d\mu(t)$ is uniform with respect to x in bounded subsets of $\mathbb{Y}$ (respectively compact subsets of $\mathbb{Y}$). The space of such functions is denoted by $\mathscr{E}\mathrm{U}_b(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$ (respectively $\mathscr{E}\mathrm{U}_c(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$).

Remark 1.1.3. *If for each $x \in \mathbb{Y}$, $f(.,x) \in \mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ and $f(t,x)$ is continuous with respect to x, uniformly with respect to t, then $f \in \mathscr{E}U_c(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$.*
Indeed, let K be a compact subset of $\mathbb{Y}$, and let $\varepsilon > 0$. Let $\mathfrak{d}$ be any distance on $\mathbb{Y}$ which generates the topology of $\mathbb{Y}$. There exists $\eta > 0$ such that, for all $x,y \in K$ satisfying $\mathfrak{d}(x,y) < \eta$, we have $\|f(t,x) - f(t,y)\| < \varepsilon$ for every $t \in \mathbb{R}$. Let $x_1,\ldots,x_m$ be a finite sequence in K such that $K \subset \cup_{i=1}^m B(x_i,\eta)$. We have, for every $r > 0$,

$$\begin{aligned}
&\sup_{x\in K}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f(t,x)\|\,d\mu(t)\\
&\qquad\le \max_{1\le i\le m}\sup_{x\in B(x_i,\eta)}\left(\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f(t,x)-f(t,x_i)\|\,d\mu(t)\right.\\
&\qquad\qquad\left.+\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f(t,x_i)\|\,d\mu(t)\right)\\
&\qquad\le\varepsilon+\max_{1\le i\le m}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|f(t,x_i)\|\,d\mu(t),
\end{aligned}$$

which shows that $f \in \mathscr{E}U_c(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$.

We say that a continuous function $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ is *μ-pseudo almost automorphic with respect to the first variable, uniformly with respect to the second variable in bounded subsets of* $\mathbb{Y}$ (respectively *in compact subsets of* $\mathbb{Y}$) if it has the form

$$\begin{aligned}
f = g+\Phi,\quad g\in \mathrm{AAU}_b(\mathbb{R}\times\mathbb{Y},\mathbb{X}),\quad \Phi\in\mathscr{E}\mathrm{U}_b(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)\\
\text{(respectively } f = g+\Phi,\quad g\in \mathrm{AAU}_c(\mathbb{R}\times\mathbb{Y},\mathbb{X}),\quad \Phi\in\mathscr{E}\mathrm{U}_c(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)).
\end{aligned}$$

The space of such functions is denoted by $\mathrm{PAAU}_b(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$ (respectively $\mathrm{PAAU}_c(\mathbb{R}\times\mathbb{Y},\mathbb{X},\mu)$).

1.1.3 Stepanov, Weyl and Besicovitch-like pseudo almost automorphy

Stepanov-like pseudo almost automorphy and variants The notion of Stepanov-like almost automorphy was proposed by Casarino in [23]. Then Stepanov-like pseudo almost automorphy was first studied by Diagana [38]. Stepanov-like weighted pseudo almost automorphy seems to have been investigated first and simultaneously in [82] and [88].
Let $p > 0$. We say that a locally p-integrable function $f : \mathbb{R} \to \mathbb{X}$ is *$\mathbb{S}^p$-almost automorphic*, or *Stepanov-like almost automorphic* if, for every sequence (t_n) in $\mathbb{R}$, there exists a subsequence (t'_n) and a locally p-integrable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty}\left\|f(t+t'_n)-g(t)\right\|_{\mathbb{S}^p}$$

and

$$\lim_{n\to\infty}\left\|g(t-t'_n)-f(t)\right\|_{\mathbb{S}^p},$$

where, for any locally p-integrable function $h:\mathbb{R}\to\mathbb{X}$,

$$\|h\|_{\mathbb{S}^p}=\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\|h(t)\|^p\,dt\right)^{1/p}.$$

The space of $\mathbb{S}^p$-almost automorphic $\mathbb{X}$-valued functions is denoted by $\mathrm{AA}_{\mathbb{S}^p}(\mathbb{R},\mathbb{X})$.
The *Bochner transform*[1] of a function $f:\mathbb{R}\to\mathbb{X}$ is the function

$$f^b:\begin{cases}\mathbb{R} & \to & \mathbb{X}^{[0,1]}\\ t & \mapsto & f(t+.).\end{cases}$$

We have

$$\mathrm{AA}_{\mathbb{S}^p}(\mathbb{R},\mathbb{X})=\left\{f;\ f^b\in\mathrm{AA}(\mathbb{R},\mathrm{L}^p([0,1],dt,\mathbb{X}))\right\}.$$

We define the Stepanov-like μ-ergodic functions in a similar way :

$$\mathscr{E}_{\mathbb{S}^p}(\mathbb{R},\mathbb{X},\mu)=\left\{f;\ f^b\in\mathscr{E}(\mathbb{R},\mathrm{L}^p([0,1],dt,\mathbb{X}),\mu)\right\}.$$

Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6). We say that $f:\mathbb{R}\to\mathbb{X}$ is $\mathbb{S}^p$*-pseudo almost automorphic*, or *Stepanov-like weighted pseudo almost automorphic* if f has the form

$$f=g+\Phi,\quad g\in\mathrm{AA}_{\mathbb{S}^p}(\mathbb{R},\mathbb{X}),\quad \Phi\in\mathscr{E}_{\mathbb{S}^p}(\mathbb{R},\mathbb{X},\mu).$$

A further extension has been imagined by Diagana [39] : it consists in adding a weight in the Stepanov norm $\|.\|_{\mathbb{S}^p}$. Let p and μ as before, and let ν be a Borel measure on the interval $[0,1]$ such that

$$0<\nu([0,1])<+\infty. \tag{1.9}$$

Set

$$\|h\|_{\mathbb{S}^p_\nu}=\sup_{x\in\mathbb{R}}\left(\int_0^1\|h(x+t)\|^p\,d\nu(t)\right)^{1/p}.$$

Then, by replacing $\|.\|_{\mathbb{S}^p}$ by $\|.\|_{\mathbb{S}^p_\nu}$, one defines in the obvious way the space $\mathrm{AA}_{\mathbb{S}^p_\nu}(\mathbb{R},\mathbb{X})$ of $\mathbb{S}^p_\nu$*-almost automorphic* $\mathbb{X}$-valued functions, the space $\mathscr{E}_{\mathbb{S}^p_\nu}(\mathbb{R},\mathrm{L}^p(\Omega,\mathrm{P},\mathbb{R}),\mu)$ of $\mathbb{S}^p_\nu$*-μ-ergodic functions*, and the space $\mathrm{PAA}_{\mathbb{S}^p_\nu}(\mathbb{R},\mathbb{X})$ of $\mathbb{S}^p_\nu$*-pseudo almost automorphic* functions.

Weyl-like and Besicovitch-like pseudo almost automorphy The concept of Weyl-like pseudo almost automorphy has been recently explored by Abbas [1]. The definition is similar to that of Stepanov-like pseudo almost automorphy, replacing $\|.\|_{\mathbb{S}^p}$ by the weaker seminorm $\|.\|_{\mathbb{W}^p}$, defined by

$$\|h\|_{\mathbb{W}^p}=\lim_{r\to+\infty}\sup_{x\in\mathbb{R}}\left(\frac{1}{2r}\int_{x-r}^{x+r}\|h(t)\|^p\,dt\right)^{1/p}.$$

A further weakening leads to the Besicovitch seminorm, which does not seem to have been investigated in the context of almost automorphy :

$$\|h\|_{\mathbb{B}^p}=\limsup_{r\to+\infty}\left(\frac{1}{2r}\int_{-r}^{r}\|h(t)\|^p\,dt\right)^{1/p}.$$

[1]The terminology is due to the fact that Bochner was the first to use this transform, for Stepanov almost periodicity, in [19].

1.1.4 Weighted pseudo almost automorphy in topological spaces

We have seen that, to define the space of almost automorphic functions, as well as that of almost periodic functions, on a space $\mathbb{X}$, there is no need to assume that $\mathbb{X}$ is a vector space, nor a metric space, these spaces depend only on the topological structure of $\mathbb{X}$. We prove in this section that the definition of $\mathrm{PAA}(\mathbb{R},\mathbb{X},\mu)$ is metric (independent of the vector structure of $\mathbb{X}$ but dependent on the metric), and that it can be made purely topological if one allows for a slight change of the definition. This leads to two topological concepts of μ-pseudo almost automorphy : in Tudor and Tudor's sense, and in the wide sense.
In this section, unless otherwise stated, $\mathbb{X}$ is only assumed to be a topological space, not necessarily metrizable.

Remark 1.1.4. *Assume that $\mathbb{X}$ is a Banach space. By definition, the space $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ consists of continuous functions $f:\mathbb{R}\to\mathbb{X}$ satisfying*

(i) f *is bounded,*

(ii) $\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|f(t)\|\,d\mu(t)=0.$

Condition (i) *is of metric nature. By [16, Theorem 2.14], Condition* (ii) *implies Condition* (1.10) *below, and the converse implication is true if we assume* (i) *:*

$$\textit{For any } \varepsilon>0,\ \lim_{r\to\infty}\frac{\mu\{t\in[-r,r];\ \|f(t)\|>\varepsilon\}}{\mu([-r,r])}=0. \tag{1.10}$$

Thus f satisfies (1.10) *if, and only if,*

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big(\|f(t)\|\wedge 1\Big)\,d\mu(t)=0.$$

Condition (1.10) *can be reformulated as*

$$\textit{For any neighbourhood } U \textit{ of } 0,\ \lim_{r\to\infty}\frac{\mu\{t\in[-r,r];\ f(t)\notin U\}}{\mu([-r,r])}=0. \tag{1.11}$$

The vector structure of $\mathbb{X}$ is still involved in (1.11) *through the vector* 0*. But we use $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ only in order to ensure that a function is close in a certain sense to $AA(\mathbb{R},\mathbb{X})$. To that end, as $AA(\mathbb{R},\mathbb{X})$ contains the constant functions, we can allow a generalization of $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ by replacing* 0 *in* (1.11) *by any other fixed point x_0 of $\mathbb{X}$.*[2]

An elegant metric definition of μ-pseudo almost periodic functions has been proposed by Constantin and Maria Tudor in [78]. If we adapt their definition, a continuous function $f:\mathbb{R}\to\mathbb{X}$ is *μ-pseudo almost automorphic in Tudor and Tudor's sense* if f has relatively compact range and there exists $g\in\mathrm{AA}(\mathbb{R},\mathbb{X})$ such that the function $t\mapsto\mathfrak{d}(f(t),g(t))$

[2] Actually, the terminology "ergodic" is misleading but it has the advantage of shortness. It would be more appropriate to follow Zhang's terminology (e.g. [85, 87]), and call *μ-ergodic perturbations* the elements of $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$, and call *ergodic* the functions f such that $1/\mu([-r,r])\int_{-r}^{r}f(t)\,d\mu(t)$ converges to some limit, not necessarily 0.

is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$. This definition is more restrictive than the standard one because the metric condition (i) is replaced by the stronger topological condition that f have relatively compact range (note that a subset A of $\mathbb{X}$ is relatively compact if, and only if, it is bounded for *every* metric which generates the topology of $\mathbb{X}$, see [41, Problem 4.3.E.(c)]).

Let us say that a continuous function $f:\mathbb{R}\to\mathbb{X}$ is *μ-pseudo almost automorphic in the wide sense* if there exists $g\in \mathrm{AA}(\mathbb{R},\mathbb{X})$ such that the function $t\mapsto \mathfrak{d}(f(t),g(t))\wedge 1$ is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$. We can get rid of the distance $\mathfrak{d}$ in this definition by using the fact that the closure of the range of any function in $\mathrm{AA}(\mathbb{R},\mathbb{X})$ is compact. On a compact space K, there is one and only one uniform structure, a basis of entourages of which consists of all sets of the form $V=\cup_{i=1}^{m}(U_i\times U_i)$, where $U_1,\dots,U_m$ is a finite open cover of K, see e.g. [22] or [41]. In this way, we obtain (ii) below, which shows that the space of functions which are μ-pseudo almost automorphic in the wide sense depends only on the topology of $\mathbb{X}$.

We have thus two possible topological definitions of μ-pseudo almost automorphy : in Tudor and Tudor's sense, and in the wide sense. The former is stronger than (1.7), while the latter is weaker.

Proposition 1.1.5. *(Topological characterization of μ-pseudo almost automorphy in the wide sense) Let $f:\mathbb{R}\to\mathbb{X}$ be continuous. Let μ be a Borel measure on $\mathbb{R}$ satisfying* (1.6). *If $\mathbb{X}$ is a metric space, the following propositions are equivalent.*

(i) f is μ-pseudo almost automorphic in the wide sense, i.e. there exists a function $g\in AA(\mathbb{R},\mathbb{X})$ such that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big(\mathfrak{d}\left(f(t),g(t)\right)\wedge 1\Big)\,d\mu(t)=0.$$

(ii) There exists a function $g\in AA(\mathbb{R},\mathbb{X})$ such that, for any finite open cover $U_1,\dots,U_m$ of the closure K of $\{g(t);t\in\mathbb{R}\}$,

$$\lim_{r\to\infty}\frac{\mu\{t\in[-r,r];\,(f(t),g(t))\notin V\}}{\mu([-r,r])}=0,$$

where $V=\cup_{i=1}^{m}(U_i\times U_i)$.

Proof : (i) $\Rightarrow$ (ii). Recall that K is compact because $g\in\mathrm{AA}(\mathbb{R},\mathbb{X})$. Let $V=\cup_{i=1}^{m}(U_i\times U_i)$ be as in (ii). Then V is an open neighborhood in $\mathbb{X}\times\mathbb{X}$ of the diagonal $\Delta=\{(x,x);x\in K\}$. Define the distance $\mathfrak{d}_2$ on $\mathbb{X}\times\mathbb{X}$ by

$$\mathfrak{d}_2\Big((x,y),(x',y')\Big)=\mathfrak{d}(x,x')+\mathfrak{d}(y,y').$$

As Δ is compact, the distance $\varepsilon=\mathfrak{d}_2(\Delta,\mathbb{X}\times\mathbb{X}\setminus V)$ is positive. Let

$$B_\varepsilon:=\{(y,z)\in\mathbb{X}\times\mathbb{X};\,\mathfrak{d}_2\Big((y,z),\Delta\Big)<\varepsilon\}\subset V.$$

For each $(y,z)\in B_\varepsilon$, there exists $x\in K$ such that $\mathfrak{d}_2\Big((y,z),(x,x)\Big)<\varepsilon$, thus

$$\mathfrak{d}(y,z)\le\mathfrak{d}(y,x)+\mathfrak{d}(x,z)<\varepsilon.$$

On the other hand, if $z \in K$ and $\mathfrak{d}(y,z) < \varepsilon$, we have

$$\mathfrak{d}_2\Big((y,z),(z,z)\Big) = \mathfrak{d}(y,z) < \varepsilon,$$

thus $(y,z) \in B_\varepsilon$. Applying this to $(f(t),g(t))$, we get, for every $r > 0$,

$$\frac{\mu\{t \in [-r,r];\, (f(t),g(t)) \notin V\}}{\mu([-r,r])} \le \frac{\mu\{t \in [-r,r];\, \mathfrak{d}(f(t),g(t)) \ge \varepsilon\}}{\mu([-r,r])}.$$

But (i) means that the function $t \mapsto \mathfrak{d}(f(t),g(t)) \wedge 1$ is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$, thus, by [16, Theorem 2.14], the latter term goes to 0 when r goes to ∞.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$. Let $U_1,\dots,U_m$ be a finite open cover of K such that $\mathrm{Diam}(U_i) > \varepsilon$, $i = 1,\dots,m$, and let $V = \cup_{i=1}^m (U_i \times U_i)$. We have

$$\frac{\mu\{t \in [-r,r];\, \mathfrak{d}(f(t),g(t)) > \varepsilon\}}{\mu([-r,r])} \le \frac{\mu\{t \in [-r,r];\, (f(t),g(t)) \notin V\}}{\mu([-r,r])},$$

where the latter term goes to 0 when r goes to ∞. The conclusion follows from [16, Theorem 2.14]. ∎

Remark 1.1.6. *The reasoning of Proposition 1.1.5 can be applied without change to give a topological characterization of μ-pseudo almost periodic functions in the wide sense. More generally, $AA(\mathbb{R},\mathbb{X})$ can be replaced in this reasoning by any class of functions which have relatively compact range.*

Theorem 1.1.1. *(Uniqueness of the decompostion of pseudo almost automorphic functions in the wide sense) Let μ be a Borel measure on $\mathbb{R}$ satisfying* (1.6) *and Condition **(H)**. Let $f : \mathbb{R} \to \mathbb{X}$ satisfying Condition* (ii) *of Proposition 1.1.5. Then the function $g \in AA(\mathbb{R},\mathbb{X})$ given by Condition* (ii) *is unique and satisfies*

$$\{g(t);\, t \in \mathbb{R}\} \subset \overline{\{f(t);\, t \in \mathbb{R}\}}.$$

Proof : Let g and K as in Condition (ii). Let $\varphi : \mathbb{X} \to \mathbb{R}$ be a continuous function. Then $\varphi \circ g \in \mathrm{AA}(\mathbb{R},\mathbb{R})$. Let $\hat{U}_1,\dots,\hat{U}_m$ be a finite open cover of the closure $\varphi(K)$ of $\{\varphi \circ g(t);\, t \in \mathbb{R}\}$, and let $\hat{V} = \cup_{i=1}^m \big(\hat{U}_i \times \hat{U}_i\big)$. Then $U_1 = \varphi^{-1}(\hat{U}_1),\dots,U_m = \varphi^{-1}(\hat{U}_m)$ form a finite open cover of K. Let $V = \cup_{i=1}^m (U_i \times U_i)$. We have

$$\frac{\mu\{t \in [-r,r];\, (\varphi \circ f(t),\varphi \circ g(t)) \notin \hat{V}\}}{\mu([-r,r])} \le \frac{\mu\{t \in [-r,r];\, (f(t),g(t)) \notin V\}}{\mu([-r,r])}.$$

Thus, by Proposition 1.1.5, $\varphi \circ f$ is in $\mathrm{PAA}(\mathbb{R},\mathbb{R},\mu)$ and the function $\varphi \circ f - \varphi \circ g$ is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$. By [16, Theorem 4.1], we have

$$\{\varphi \circ g(t);\, t \in \mathbb{R}\} \subset \overline{\{\varphi \circ f(t);\, t \in \mathbb{R}\}},$$

and, by the uniqueness of the decomposition of vector-valued μ-pseudo almost automorphic functions [16, Theorem 4.7] if $g' \in \mathrm{AA}(\mathbb{R},\mathbb{X})$ satisfies the same condition as g, we have $\varphi \circ g' = \varphi \circ g$. As φ is arbitrary, we deduce $g' = g$.

∎

1.1.5 Weighted pseudo almost automorphy for stochastic processes

From now on, $\mathbb{X}$ and $\mathbb{Y}$ are assumed to Polish spaces, i.e. separable metrizable topological spaces whose topology is generated by a complete metric.

1.1.6 Weighted pseudo almost automorphy in pth mean

We assume here that $\mathbb{X}$ is a Banach space. Let $X = (X_t)_{t\in\mathbb{R}}$ be a continuous stochastic process with values in $\mathbb{X}$, defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$. Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6).
Let $p > 0$. We say that X is *almost automorphic in pth mean* (respectively *μ-pseudo almost automorphic in pth mean*) if the mapping $t \mapsto X(t)$ is in $\mathrm{AA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))$ (respectively in $\mathrm{PAA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu)$, i.e., if it has the form $X = Y + Z$, where $Y \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))$ and $Z \in \mathscr{E}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu)$). When $p = 2$, we say that X is *square-mean almost automorphic* (respectively *square-mean μ-pseudo almost automorphic*).
The process X is said to be *almost automorphic in probability* if the mapping $X : t \to \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ is almost automorphic, where $\mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ is the space of measurable mappings from Ω to $\mathbb{X}$, endowed with the topology of convergence in probability. Recall that the topology of $\mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ is induced by e.g. the distance

$$\mathfrak{d}_{\text{Prob}}(U, V) = \mathrm{E}\left(\|U - V\| \wedge 1\right),$$

which is complete.
The process X is said to be *μ-pseudo almost automorphic in probability*, and we write $X \in \mathrm{PAA}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X}), \mu)$, if the mapping $t \mapsto X(t)$, $\mathbb{R} \to \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ is μ-pseudo almost automorphic in the wide sense (or, equivalently, if it is μ-pseudo almost automorphic when $\mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ is endowed with $\mathfrak{d}_{\text{Prob}}$), i.e. if it has the form $X = Y + Z$ where $Y \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X}))$ and Z satisfies

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \mathrm{E}\left(\|Z(t)\| \wedge 1\right) d\mu(t) = 0. \tag{1.12}$$

We denote by $\mathscr{E}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X}), \mu)$ the set of stochastic processes Z satisfying (1.32).
Note that, for $p \geq 0$, the previous decompositions of $X \in \mathrm{PAA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu)$ are unique under the condition **(H)**.
Clearly, for $0 \leq p \leq q$, we have

$$\mathrm{AA}(\mathbb{R}, \mathrm{L}^q(\Omega, \mathrm{P}, \mathbb{X})) \subset \mathrm{AA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))$$

and

$$\mathrm{PAA}(\mathbb{R}, \mathrm{L}^q(\Omega, \mathrm{P}, \mathbb{X}), \mu) \subset \mathrm{PAA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu).$$

Conversely, if the set $\{\|X(t)\|^q; t \in \mathbb{R}\}$ is uniformly integrable, we have the implications

$$\left(X \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))\right) \Rightarrow \left(X \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^q(\Omega, \mathrm{P}, \mathbb{X}))\right),$$
$$\left(X \in \mathrm{PAA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu)\right) \Rightarrow \left(X \in \mathrm{PAA}(\mathbb{R}, \mathrm{L}^q(\Omega, \mathrm{P}, \mathbb{X}), \mu)\right).$$

A process X is *Stepanov-like almost automorphic in pth mean* if X is in $\mathrm{AA}_{\mathbb{S}^p}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))$. We define in the same way the processes which are *Stepanov-like μ-pseudo almost automorphic in pth mean, Weyl-like (μ-pseudo) almost automorphic in pth mean, Besicovitch-like (μ-pseudo) almost automorphic in pth mean*.

1.1.7 Weighted pseudo almost automorphy in distribution

We denote by $\mathrm{law}(X)$ the law (or distribution) of a random variable X. For any topological space $\mathbb{X}$, we denote by $\mathscr{M}^{1,+}(\mathbb{X})$ the set of Borel probability measures on $\mathbb{X}$, endowed with the topology of narrow (or weak) convergence, i.e. the coarsest topology such that the mappings $\mu \mapsto \mu(\varphi)$, $\mathscr{M}^{1,+}(\mathbb{X}) \to \mathbb{R}$ are continuous for all bounded continuous $\varphi : \mathbb{X} \to \mathbb{R}$.
If $\tau : \mathbb{X} \to \mathbb{Y}$ is a Borel measurable mapping and μ is a Borel measure on $\mathbb{X}$, we denote by $\tau_\sharp \mu$ the Borel measure on $\mathbb{Y}$ defined by

$$\tau_\sharp \mu(B) = \mu(\tau^{-1}(B))$$

for every Borel set of $\mathbb{Y}$.
Let $\mathrm{BC}(\mathbb{X}, \mathbb{R})$ denote the space of bounded continuous functions from $\mathbb{X}$ to $\mathbb{R}$, which we endow with the norm

$$\|\varphi\|_\infty = \sup_{x \in \mathbb{X}} |\varphi(x)|\,.$$

For a given distance $\mathfrak{d}$ on $\mathbb{X}$, and for $\varphi \in \mathrm{BC}(\mathbb{X}, \mathbb{R})$ we define

$$\|\varphi\|_{\mathrm{L}} = \sup\Big\{ \frac{\varphi(x) - \varphi(y)}{\mathfrak{d}(x,y)}; x \neq y \Big\}$$
$$\|\varphi\|_{\mathrm{BL}} = \max\{\|\varphi\|_\infty, \|\varphi\|_{\mathrm{L}}\}.$$

We denote

$$\mathrm{BL}(\mathbb{X}, \mathbb{R}) = \big\{\varphi \in \mathrm{BC}(\mathbb{X}, \mathbb{R}); \|\varphi\|_{\mathrm{BL}} < \infty\big\}.$$

The *bounded Lipschitz distance* $\mathfrak{d}_{\mathrm{BL}}$ associated with $\mathfrak{d}$ on $\mathscr{M}^{1,+}(\mathbb{X})$ is defined by

$$\mathfrak{d}_{\mathrm{BL}}(\mu, \nu) = \sup_{\substack{\varphi \in \mathrm{BL}(\mathbb{X}, \mathbb{R}) \\ \|\varphi\|_{\mathrm{BL}} \leq 1}} \int_{\mathbb{X}} \varphi \, d(\mu - \nu).$$

This metric generates the narrow (or weak) topology on $\mathscr{M}^{1,+}(\mathbb{X})$.
Let $p \geq 0$, and let (X_n) be a sequence in $\mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X})$. We say that (X_n) *converges in p-distribution* (or simply *converges in distribution* if $p = 0$) to a random vector X if

(i) the sequence $(\mathrm{law}(X_n))$ converges to $\mathrm{law}(X)$ for the narrow topology on $\mathscr{M}^{1,+}(\mathbb{X})$,

(ii) if $p > 0$, the sequence $(\|X_n\|^p)$ is uniformly integrable.

Almost automorphy in distribution If X is continuous with values in $\mathbb{X}$, we denote by $\widetilde{X}(t)$ the random variable $X(t+.)$ with values in $\mathrm{C}(\mathbb{R},\mathbb{X})$.
We say that X is *almost automorphic in one-dimensional distributions* if the mapping $t \mapsto \mathrm{law}(X(t))$, $\mathbb{R} \mapsto \mathscr{M}^{1,+}(\mathbb{X})$ is almost automorphic.

Remark 1.1.7. *One-dimensional distributions of a process reflect poorly its properties. For example, consider the Ornstein-Uhlenbeck process*

$$X(t) = \sqrt{2\alpha\sigma}\int_{-\infty}^{t} e^{-\alpha(t-s)}dW(s),$$

and set $Y(t) = X(0)$, $t \in \mathbb{R}$. The processes X and Y have the same one-dimensional distributions with completely different trajectories and behaviors. The trajectories of Y are constant, whereas the covariance $\mathrm{Cov}(X(t+\tau),X(t))$ converges to 0 when τ goes to ∞.

We say that X is *almost automorphic in finite dimensional distributions* if, for every finite sequence $t_1,\ldots,t_m \in \mathbb{R}$, the mapping $t \mapsto \mathrm{law}(X(t+t_1),\ldots,X(t+t_m))$, $\mathbb{R} \mapsto \mathscr{M}^{1,+}(\mathbb{X}^m)$ is almost automorphic.
We say that X is *almost automorphic in distribution* if the mapping $t \mapsto \mathrm{law}(\widetilde{X}(t))$, $\mathbb{R} \mapsto \mathscr{M}^{1,+}(\mathrm{C}_k(\mathbb{R},\mathbb{X}))$ is almost automorphic, where $\mathrm{C}_k(\mathbb{R},\mathbb{X})$ denotes the space $\mathrm{C}(\mathbb{R},\mathbb{X})$ endowed with the topology of uniform convergence on compact subsets. The Ornstein-Uhlenbeck process

$$X(t) = \sqrt{2\alpha\sigma}\int_{-\infty}^{t} e^{-\alpha(t-s)}dW(s)$$

is almost automorphic in distribution because the mapping $t \mapsto \mathrm{law}(\widetilde{X}(t))$ is constant.

Remark 1.1.8. *If X is a deterministic process $\mathbb{R} \to \mathbb{X}$, we have the equivalences*

$$\begin{aligned} X \in AA(\mathbb{R},\mathbb{X}) \Leftrightarrow & X \textit{ is almost automorphic in one-dimensional distributions} \\ \Leftrightarrow & X \textit{ is almost automorphic in finite dimensional distributions} \end{aligned}$$

and

$$X \in AA_c(\mathbb{R},\mathbb{X}) \Leftrightarrow X \textit{ is almost automorphic in distribution}.$$

Actually, as Remark 1.1.8 suggests, the definition of almost automorphy in distribution implies a stronger property for the mapping $t \mapsto \mathrm{law}(\widetilde{X}(t))$. We need first some notations. For simplicity, we set $\mathrm{C}_k = \mathrm{C}_k(\mathbb{R},\mathbb{X})$. For every $t \in \mathbb{R}$, we define a continuous operator on C_k :

$$\tau_t : \begin{cases} \mathrm{C}_k & \to \quad \mathrm{C}_k \\ x & \mapsto \quad x(t+.) = \widetilde{x}(t). \end{cases}$$

Proposition 1.1.9. *If X is almost automorphic in distribution, the mapping $t \mapsto \mathrm{law}(\widetilde{X}(t))$ is in $AA_c(\mathbb{R},\mathscr{M}^{1,+}(\mathrm{C}_k(\mathbb{R},\mathbb{X})))$. Furthermore, for any sequence (t_n) in $\mathbb{R}$ such that, for every $t \in \mathbb{R}$, $(\mathrm{law}(\widetilde{X}(t+t_n)))$ converges to a limit $g(t) \in \mathscr{M}^{1,+}(\mathrm{C}_k)$, the function g satisfies, for every $t \in \mathbb{R}$, the consistency relation*

$$g(t) = (\tau_t)_\sharp\, g(0). \tag{1.13}$$

Proof : Let us denote $f(t) = \text{law}(\widetilde{X}(t))$. We have, for every $t \in \mathbb{R}$,

$$f(t) = \text{law}(\tau_t \circ X) = (\tau_t)_\sharp f(0).$$

Let (t_n) be a sequence in $\mathbb{R}$ such that, for each $t \in \mathbb{R}$, the sequence $(f(t+t_n))$ converges to some $g(t) \in \mathscr{M}^{1,+}(\mathrm{C}_k)$. Then, for every $t \in \mathbb{R}$, (1.13) is satisfied by continuity of the operator $(\tau_t)_\sharp$. To prove the continuity of g, let us endow C_k with the distance

$$\underline{\mathfrak{d}}(x,y) = \sum_{k\geq 1} 2^{-k} \sup_{-k\leq t\leq k} \Big(\mathfrak{d}\big(x(t),y(t)\big) \wedge 1\Big), \tag{1.14}$$

and let $\underline{\mathfrak{d}}_{\text{BL}}$ be the associated bounded Lipschitz distance on $\mathscr{M}^{1,+}(\mathrm{C}_k)$. Let us show that the convergence of $(f(.+t_n))$ is uniform for $\underline{\mathfrak{d}}_{\text{BL}}$ on compact intervals. Let $r \geq 1$ be an integer. For every $t \in [-r,r]$, and for all $x,y \in \mathrm{C}_k$, we have

$$\begin{aligned}
\underline{\mathfrak{d}}(\tau_t(x),\tau_t(y)) &= \sum_{k\geq 1} 2^{-k} \sup_{-k\leq s\leq k} \Big(\mathfrak{d}\big(x(s+t),y(s+t)\big) \wedge 1\Big)\\
&\leq \sum_{k\geq 1} 2^{-k} \sup_{-k-r\leq s\leq k+r} \Big(\mathfrak{d}\big(x(s),y(s)\big) \wedge 1\Big)\\
&\leq 2^r \underline{\mathfrak{d}}(x,y).
\end{aligned}$$

Thus, for any 1-Lipschitz mapping $\varphi : \mathrm{C}_k \to \mathbb{R}$, the mapping $\varphi \circ \tau_t$ is 2^r-Lipschitz. We deduce that, if $\|\varphi\|_{\text{BL}} \leq 1$, we have $\|\varphi \circ \tau_t\|_{\text{BL}} \leq 1 + 2^r$. We have thus

$$\begin{aligned}
\underline{\mathfrak{d}}_{\text{BL}}\big(f(t+t_n), f(t+t_{n+m})\big) &= \sup_{\|\varphi\|_{\text{BL}}\leq 1} \mathrm{E}\Big(\varphi \circ \tau_t \circ \widetilde{X}(t_n) - \varphi \circ \tau_t \circ \widetilde{X}(t_{n+m})\Big)\\
&\leq (1+2^r) \sup_{\|\varphi\|_{\text{BL}}\leq 1} \mathrm{E}\Big(\varphi \circ \widetilde{X}(t_n) - \varphi \circ \widetilde{X}(t_{n+m})\Big)\\
&= (1+2^r)\underline{\mathfrak{d}}_{\text{BL}}\big(f(t_n), f(t_{n+m})\big),
\end{aligned}$$

which shows that $(f(.+t_n))$ is uniformly Cauchy on $[-r,r]$. Thus g is continuous, and $f \in \mathrm{AA}_c(\mathbb{R}, \mathscr{M}^{1,+}(\mathrm{C}_k(\mathbb{R},\mathbb{X})))$. ■

We denote by
- $\mathrm{AAD}_1(\mathbb{R},\mathbb{X})$ the set of $\mathbb{X}$-valued processes which are almost automorphic in one-dimensional distributions,
- $\mathrm{AAD}_f(\mathbb{R},\mathbb{X})$ the set of $\mathbb{X}$-valued processes which are almost automorphic in finite dimensional distributions,
- $\mathrm{AAD}(\mathbb{R},\mathbb{X})$ the set of $\mathbb{X}$-valued processes which are almost automorphic in distribution.

(In these notations, we omit the probability space $(\Omega,\mathscr{F},\mathrm{P})$, as there is no ambiguity here.) We have the inclusions

$$\mathrm{AAD}(\mathbb{R},\mathbb{X}) \subset \mathrm{AAD}_f(\mathbb{R},\mathbb{X}) \subset \mathrm{AAD}_1(\mathbb{R},\mathbb{X}).$$

The following result is in the line of [10, Theorem 2.3].

Theorem 1.1.2. *Let X be an $\mathbb{X}$-valued stochastic process, and let $\mathfrak{d}$ be a distance on $\mathbb{X}$ which generates the topology of $\mathbb{X}$. Asume that X satisfies the tightness condition*

$$\forall [a,b]\subset\mathbb{R}, \forall\varepsilon>0,\ \forall\eta>0,\ \exists\delta>0, \forall r\in\mathbb{R},\\ \mathrm{P}\Big\{\sup_{\substack{|t-s|<\delta\\ t,s\in[a,b]}}\mathfrak{d}\left(X(r+t),X(r+s)\right)>\eta\Big\}<\varepsilon. \quad (1.15)$$

Then the following properties are equivalent :

(a) $X\in AAD_f(\mathbb{R},\mathbb{X})$.

(b) $X\in AAD(\mathbb{R},\mathbb{X})$.

Proof : Clearly (b)⇒(a). Assume that $X\in \mathrm{AAD}_f(\mathbb{R},\mathbb{X})$. Let (γ'_n) be a sequence in $\mathbb{R}$, and, for $t_1,t_2,\dots,t_k,t\in\mathbb{R}$ define (using notations of [77])

$$\mu_t^{t_1,\dots,t_k} := \mathrm{law}(X(t_1+t),\dots,X(t_k+t)).$$

By a diagonal procedure we can find a subsequence (γ_n) of (γ'_n) such that, for every $k\geq 1$, for all $q_1,q_2,\dots,q_k\in\mathbb{Q}\bigcap\mathbb{R}$ (where $\mathbb{Q}$ is the set of rational numbers), and for every $t\in\mathbb{R}$,

$$\lim_n\lim_m \mu_{t+\gamma_n-\gamma_m}^{q_1,\dots,q_k} = \mu_t^{q_1,\dots,q_k}.$$

Let $\mathfrak{d}_k$ be the distance on $\mathbb{X}^k$ defined by

$$\mathfrak{d}_k\big((x_1,\dots,x_k),(y_1,\dots,y_k)\big) = \max_{1\leq i\leq k}\mathfrak{d}(x_i,y_i),$$

and let $\mathfrak{d}_{\mathrm{BL}}$ the associated bounded Lipschitz distance on $\mathscr{M}^{1,+}\left(\mathbb{X}^k\right)$. We have, for all $t_1,t_2,\dots,t_k,t\in\mathbb{R}$, for all $q_1,q_2,\dots,q_k\in\mathbb{Q}\bigcap\mathbb{R}$, and for all $n,m\in\mathbb{N}$,

$$\begin{aligned}\mathfrak{d}_{\mathrm{BL}}\left(\mu_{t+\gamma_n-\gamma_m}^{q_1,\dots,q_k},\mu_{t+\gamma_n-\gamma_m}^{t_1,\dots,t_k}\right) &= \sup_{\|f\|_{\mathrm{BL}}\leq 1}\int_{\mathbb{X}^k} f\,d\left(\mu_{t+\gamma_n-\gamma_m}^{q_1,\dots,q_k}-\mu_{t+\gamma_n-\gamma_m}^{t_1,\dots,t_k}\right)\\ &\leq \max_{1\leq i\leq k}\int_\Omega \mathfrak{d}\Big(X(q_i+t+\gamma_n-\gamma_m),X(t_i+t+\gamma_n-\gamma_m)\Big)\,d\,\mathrm{P},\end{aligned}$$

so that, by (1.15), if $(q_1,\dots,q_k)\to(t_1,\dots,t_k)$, then

$$\mathfrak{d}_{\mathrm{BL}}\left(\mu_{t+\gamma_n-\gamma_m}^{q_1,\dots,q_k},\mu_{t+\gamma_n-\gamma_m}^{t_1,\dots,t_k}\right)\to 0$$

uniformly with respect to $t\in\mathbb{R}$ and $n,m\in\mathbb{N}$. By a classical result on inversion of limits, we deduce that, for all $k\geq 1$ and $t_1,\dots,t_k,t\in\mathbb{R}$,

$$\lim_n\lim_m \mu_{t+\gamma_n-\gamma_m}^{t_1,\dots,t_k} = \mu_t^{t_1,\dots,t_k}.$$

Therefore, to show that

$$\lim_n\lim_m \mathrm{law}(\widetilde{X}(t+\gamma_n-\gamma_m)) = \mathrm{law}(\widetilde{X}(t)),$$

it is enough to prove that $(\widetilde{X}(t))_{t\in\mathbb{R}}$ is tight in $C_k(\mathbb{R},\mathbb{X})$. Since $X \in \mathrm{AAD}_f(\mathbb{R},\mathbb{X})$, the family $(X(t))_{t\in\mathbb{R}} = (\widetilde{X}(t)(0))_{t\in\mathbb{R}}$ is tight, by Prokhorov's theorem for relatively compact sets of probability measures on Polish spaces. By (1.15) and the Arzelà-Ascoli-type characterization of tight subsets of $\mathscr{M}^{1,+}(\mathbb{X})$ (see e.g. the proof of [13, Theorem 7.3] or [81, Theorem 4]), we conclude that $(\widetilde{X}(t))_{t\in\mathbb{R}}$ is tight in $C_k(\mathbb{R},\mathbb{X})$, which proves our claim. ∎

Remark 1.1.10. *Assume that $\mathbb{X}$ is a vector space. The spaces $AAD(\mathbb{R},\mathbb{X})$, $AAD_f(\mathbb{R},\mathbb{X})$, and $AAD_1(\mathbb{R},\mathbb{X})$ are not vector spaces. Indeed, the Ornstein-Uhlenbeck process*

$$X(t) = \sqrt{2\alpha\sigma}\int_{-\infty}^{t} e^{-\alpha(t-s)}dW(s)$$

. For each $t \in \mathbb{R}$, let $Y(t) = X(0)$. The processes X and Y are stationary in the strong sense, thus they are in $AAD(\mathbb{R},\mathbb{X})$. For each $t \in \mathbb{R}$, the variable $Z(t) = X(t)+Y(t)$ is Gaussian centered with variance

$$\begin{aligned}\operatorname{Var} Z(t) &= \mathrm{E}\left(X^2(t)\right) + \mathrm{E}\left(Y^2(t)\right) + 2\operatorname{Cov}\big(X(t),Y(t)\big)\\ &= 2\sigma^2 + 2\sigma^2\exp\left(-\alpha\,|t|\right)\\ &\to 2\sigma^2 \textit{ when } |t| \to \infty.\end{aligned}$$

Thus $\mathrm{law}(Z(t))$ *is the Gaussian distribution $\mathscr{N}(0,2\sigma^2(1+\exp(-\alpha\,|t|)))$, which converges when $|t| \to \infty$ to $\mathscr{N}(0,2\sigma^2)$. Set $\mathfrak{m}(t) = \mathscr{N}(0,2\sigma^2)$, $t \in \mathbb{R}$. For each $t \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} \mathrm{law}(Z(t+n)) = \mathfrak{m}(t)$$
$$\lim_{n\to\infty} \mathfrak{m}(t-n) = \mathfrak{m}(t) \neq \mathrm{law}(Z(t)).$$

Thus $Z \notin AAD_1(\mathbb{R},\mathbb{X})$.
This contradicts [45, Lemma 2.3].

Almost automorphy in p-distribution A useful variant of almost automorphy in distribution takes into account integrability of order p. Let $p \geq 0$. We say that a continuous $\mathbb{X}$-valued stochastic process is *almost automorph in p-distribution* if

(i) $X \in \mathrm{AAD}(\mathbb{R},\mathbb{X})$,

(ii) if $p > 0$, the family $(\|X(t)\|^p)_{t\in\mathbb{R}}$ is uniformly integrable.

These conditions imply that the mapping $t \mapsto X(t)$, $\mathbb{R} \to \mathrm{L}^p(\Omega,\mathrm{P},\mathbb{X})$, is continuous.
We denote by $\mathrm{AAD}^p(\mathbb{R},\mathbb{X})$ the set of $\mathbb{X}$-valued processes which are almost automorphic in p-distribution, in particular we have $\mathrm{AAD}^0(\mathbb{R},\mathbb{X}) = \mathrm{AAD}(\mathbb{R},\mathbb{X})$. Similarly, for $p \geq 0$, one defines the sets $\mathrm{AAD}^p_f(\mathbb{R},\mathbb{X})$ and $\mathrm{AAD}^p_1(\mathbb{R},\mathbb{X})$ of processes which are respectively *almost automorphic in one-dimensional p-distributions* and *almost automorphic in finite dimensional p-distributions*.

Weighted pseudo almost automorphy in distribution and variants As usual, we assume that μ is a Borel measure on $\mathbb{R}$ satisfying (1.6).
Tudor and Tudor proposed in [78] a very natural and elegant notion of pseudo almost periodicity in (one-dimensional) distribution that can easily be extended to weighted μ-pseudo almost automorphy : X is *μ-pseudo almost periodic in one-dimensional distributions in Tudor and Tudor's sense* if it satisfies

(TT_1) The mapping $t \mapsto \mathrm{law}(X(t))$ is continuous with relatively compact range in $\mathscr{M}^{1,+}(\mathbb{X})$, and there exists an almost automorphic function $\mathfrak{m} : \mathbb{R} \to \mathscr{M}^{1,+}(\mathbb{X})$ such that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathfrak{d}_{\mathrm{BL}}(\mathrm{law}(X(t)), \mathfrak{m}(t)) d\mu(t) = 0. \tag{1.16}$$

Similar definitions are easy to write for μ-pseudo almost automorphy in finite distributions or in distribution.
Recall that $\mathfrak{d}_{\mathrm{BL}}$ is bounded. If we remove the condition of relatively compact range (as in Proposition 1.1.5), we get three distributional notions of *μ-pseudo almost automorphy in the wide sense : in one-dimensional distributions, in finite dimensional distributions, and in distribution.*
We propose three stronger notions of μ-pseudo almost automorphy in a distributional sense, that seem to be particularly useful for stochastic equations.
Assume that $\mathbb{X}$ is a vector space. Let $p \geq 0$. We say that X is *μ-pseudo almost automorphic in p-distribution* if X can be written

$$X = Y + Z, \text{ where } Y \in \mathrm{AAD}^p(\mathbb{R}, \mathbb{X}) \text{ and } Z \in \mathscr{E}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu).$$

The set of $\mathbb{X}$-valued processes which are μ-pseudo almost automorphic in p-distribution is denoted by $\mathrm{PAAD}^p(\mathbb{R}, \mathbb{X})$. Similar definitions hold for the spaces $\mathrm{PAAD}_1^p(\mathbb{R}, \mathbb{X}, \mu)$ and $\mathrm{PAAD}_f^p(\mathbb{R}, \mathbb{X}, \mu)$ of processes which are *μ-pseudo almost automorphic in one-dimensional p-distributions* and *in finite dimensional p-distributions* respectively.

Remark 1.1.11. *The definitions we propose for μ-pseudo almost automorphy in distribution, or in finite dimensional distributions, or in one-dimensional distributions, are in a way stronger and less natural than those in the wide distributional sense, because they involve (at least, apparently) not only the distribution of the process, but the probability space* $(\Omega, \mathscr{F}, \mathrm{P})$. *For example, a random process X is in $PAAD_1^0(\mathbb{R}, \mathbb{X}, \mu)$ if, and only if, there exists a process Y defined on the same probability space such that* $\mathfrak{m}(.) := \mathrm{law}(Y(.))$ *is in* $AA(\mathbb{R}, \mathscr{M}^{1,+}(\mathbb{X}))$ *and satisfies* (1.16). *Note also that, in our definition, the ergodic part is ergodic in pth mean. Our definitions are thus intermediate between μ-pseudo almost automorphy in a purely distributional sense and μ-pseudo almost automorphy in pth mean. However, our definitions seem to be convenient for calculations, and Theorem 4.1.1 shows that, for some stochastic differential equations with μ-pseudo almost automorphic coefficients, the process Y appears naturally : it is the solution of the corresponding SDE where the coefficients are the almost automorphic parts of the coefficients of the original SDE.*

1.1.8 Pseudo-almost automorphy in p-mean vs in p-distribution

Let $X = (X_t)_{t\in\mathbb{R}}$ be a continuous stochastic process with values in $\mathbb{X}$, defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$. Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6). Clearly, we have for all $p \geq 0$,

$$\left(X \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))\right) \Rightarrow \left(X \in \mathrm{AAD}_f^p(\mathbb{R}, \mathbb{X})\right).$$

Using Theorem 1.1.2, we can get more : if X satisfies (1.15), we deduce, for every $p \geq 0$,

$$\left(X \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}))\right) \Rightarrow \left(X \in \mathrm{AAD}^p(\mathbb{R}, \mathbb{X})\right),$$
$$\left(X \in \mathrm{PAA}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu)\right) \Rightarrow \left(X \in \mathrm{PAAD}^p(\mathbb{R}, \mathbb{X}, \mu)\right).$$

The converse implications are false. Indeed, Example 3.0.4 shows that a process which is almost automorphic in distribution is not necessarily almost automorphic in probability or in p-mean, see also [10, Counterexample 2.16][3].
The same counterexample also shows that a process which is μ-pseudo almost automorphic in p-distribution is not necessarily μ-pseudo almost automorphic in probability or in p-mean.

1.2 Stepanov Almost periodicity and its variants in metric space

In this section, we present the concept of Stepanov (μ-pseudo) almost periodic function and related concepts like almost periodicity in Lebesgue measure. Moreover, we also recall some useful and key results. We begin with some notations.

1.2.1 Notations

In what follows, $(\mathbb{E}, d)$ is a complete metric space. If $\mathbb{X}$ and $\mathbb{Y}$ are two metric spaces, we indistinctly denote by $\mathfrak{d}$ a distance on $\mathbb{X}$ (respectively on $\mathbb{Y}$). When $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, their norms are denoted by $\|.\|$, and $\mathfrak{d}$ is assumed to result from $\|.\|$. We denote by $\mathrm{C}(\mathbb{X}, \mathbb{Y})$ the space of continuous functions from $\mathbb{X}$ to $\mathbb{Y}$. When this space is endowed with the topology of uniform convergence on compact subsets of $\mathbb{X}$, it is denoted by $\mathrm{C}_k(\mathbb{X}, \mathbb{Y})$.

1.2.2 Stepanov and Bohr almost periodicity

Let us recall some definitions of (Stepanov) almost periodic functions and some keys results.
Stepanov almost periodicity Following [30,33], let $M(\mathbb{R}, \mathbb{E})$ be the class of measurable functions from $\mathbb{R}$ to $\mathbb{E}$. Let $p \geq 1$. We fix a point x_0 in $\mathbb{E}$. We denote by $\mathfrak{L}^p(\mathbb{R}, \mathbb{E})$, the

[3] Let us here point out an infortunate error in [60] : it is mistakenly said at the end of Section 1 of [60] that almost periodicity in square mean implies almost periodicity in distribution, and that the converse is true under a tightness condition. The first claim is true under the tightness condition (1.15), whereas Example 3.0.4, which is also Example 2.1 of [60], disproves the second claim.

subset of $M(\mathbb{R},\mathbb{E})$ of locally p-integrable functions, that is,

$$\mathfrak{L}^p(\mathbb{R},\mathbb{E}) = \left\{ f \in M(\mathbb{R},\mathbb{E}),\ \text{for any } a,b \in \mathbb{R};\ \int_{[a,b]} d^p\left(f(t),x_0\right) dt < +\infty \right\}.$$

Define $\mathrm{L}^p(0,1;\mathbb{E})$ as the class

$$\mathrm{L}^p(0,1;\mathbb{E}) = \left\{ f \in M(\mathbb{R},E),\ \int_{[0,1]} d^p\left(f(t),x_0\right) dt < +\infty \right\}$$

which is a complete metric space, when it is endowed with the metric

$$\mathscr{D}_{\mathrm{L}^p}(f,g) = \left(\int_{[0,1]} d^p(f(t),g(t))dt \right)^{1/p}.$$

We denote by $\mathrm{L}^\infty(\mathbb{R},\mathbb{E})$ the space of all $\mathbb{E}$-valued essentially bounded functions, endowed with essential supremum metric. Obviously, all the previous spaces do not depend on the choice of the point $x_0 \in \mathbb{E}$.

Recall that a set $A \subset \mathbb{R}$ is *relatively dense* if there exists a real number $\ell > 0$, such that $A \cap [a,a+\ell] \neq \emptyset$, for all a in $\mathbb{R}$.

We say that a locally p-integrable function $f : \mathbb{R} \to \mathbb{E}$ is *Stepanov almost periodic of order p* or *$\mathbb{S}^p$-almost periodic*, if, for all $\varepsilon > 0$, the set

$$\mathbb{S}^p T(f,\varepsilon) := \left\{ \tau \in \mathbb{R},\ D^d_{\mathbb{S}^p}\left(f(.+\tau), f(.)\right) \leq \varepsilon \right\}$$

is relatively dense, where, for any locally p-integrable functions $f,g : \mathbb{R} \to \mathbb{E}$,

$$D^d_{\mathbb{S}^p}(f,g) = \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} d^p(f(t),g(t))\, dt \right)^{1/p}.$$

The space of $\mathbb{S}^p$-almost periodic $\mathbb{E}$-valued functions is denoted by $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$ (or $\mathbb{S}\mathrm{AP}(\mathbb{R})$ when $\mathbb{E} = \mathbb{R}$ and $p = 1$). Let $\mathbb{S}^\infty\mathrm{AP}(\mathbb{R},\mathbb{E})$ be the space of functions $f \in \mathrm{L}^\infty(\mathbb{R},\mathbb{E})$ such that for any $\varepsilon > 0$, there exists a relatively dense $T^d_{\mathrm{L}^\infty}(f,\varepsilon)$ such that

$$\mathscr{D}^d_{\mathrm{L}^\infty}\left(f(.+\tau), f(.)\right) \leq \varepsilon, \text{ for all } \tau \in T^d_{\mathrm{L}^\infty}(f,\varepsilon).$$

The relation $\lim_{p\to\infty} D^d_{\mathbb{S}^p}(f,g) = D^d_{\mathrm{L}^\infty}(f,g)$ holds for any functions $f,g \in M(\mathbb{R},\mathbb{E})$, see [6] for the proof.

As in the case of Bochner-almost periodic functions, we have similar characterizations of Stepanov almost periodic functions. Let $f \in \mathfrak{L}^p(\mathbb{R},\mathbb{E})$, $p \in [1,+\infty[$. Then, the following statements are equivalent (compare with [48, Theorem 1]) :

– f is $\mathbb{S}^p$-almost periodic.
– f is $\mathbb{S}^p$-almost periodic in Bochner sense, that is, from every real sequence $(\alpha'_n) \subset \mathbb{R}$ one can extract a subsequence (α_n) of (α'_n) and there exists a function $g \in \mathfrak{L}^p(\mathbb{R},\mathbb{E})$ such that

$$\lim_{n\to+\infty} D^d_{\mathbb{S}_p}\left(f(.+\alpha_n), g(.)\right) = 0.$$

– f satisfies *Bochner's type double sequence criterion*, that is, for every pair of sequences $\{\alpha_n'\} \subset \mathbb{R}$ and $\{\beta_n'\} \subset \mathbb{R}$, there are subsequences $(\alpha_n) \subset (\alpha_n')$ and $(\beta_n) \subset (\beta_n')$ respectively with same indexes such that, for every $t \in \mathbb{R}$, the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+\alpha_n+\beta_m) \text{ and } \lim_{n\to\infty} f(t+\alpha_n+\beta_n), \tag{1.17}$$

exist and are equal, in the sense of the L^p-metric

$$\mathscr{D}^d_{\mathrm{L}^p}(h(t+.),g(t+.)) = \left(\int_{[0,1]} d^p(h(t+s),g(t+s))\,ds\right)^{1/p}$$

for $h,g \in \mathfrak{L}^p(\mathbb{R},\mathbb{E})$.

The proof of these equivalences follows from the fact that the concept of Stepanov almost periodicity can be seen as Bohr almost periodicity of some function with values in the Lebesgue space $\mathrm{L}^p(0,1;\mathbb{E})$. More precisely, one defines the *Bochner transform* [19] of a function $f \in \mathfrak{L}^p(\mathbb{R},\mathbb{E})$ as follows

$$f^b : \begin{cases} \mathbb{R} & \to & \mathbb{E}^{[0,1]} \\ t & \mapsto & f(t+.). \end{cases}$$

Then $f \in \mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$ if, and only if, $f^b \in \mathrm{AP}(\mathbb{R},\mathrm{L}^p(0,1;\mathbb{E}))$, and the previous equivalences become a simple consequence of the fact that $f_n \to f$ if and only if $f_n^b \to f^b$ (see e.g. [2,4,8,52]).

Since functions in $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$ are bounded with respect to the Stepanov metric, one denotes by $\mathbb{S}^p(\mathbb{R},\mathbb{E})$ (or $\mathbb{S}^p(\mathbb{R})$ when $\mathbb{E}=\mathbb{R}$, and $\mathbb{S}(\mathbb{R},\mathbb{E})$ when $p=1$) the set of all $D^d_{\mathbb{S}^p}$-bounded functions, that is, for some (or any) fixed $x_0 \in \mathbb{E}$

$$\mathbb{S}^p(\mathbb{R},\mathbb{E}) = \{f \in M(\mathbb{R},\mathbb{E}); D^d_{\mathbb{S}^p}(f,x_0) < +\infty\}.$$

So from now on, the space $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$ will be seen as a (closed) subset of the complete metric space $(\mathbb{S}^p(\mathbb{R},\mathbb{E}), D^d_{\mathbb{S}^p})$. We have the following inclusions :

$$\mathrm{AP}(\mathbb{R},\mathbb{E}) \subset \mathbb{S}^\infty\mathrm{AP}(\mathbb{R},\mathbb{E}) \subset \mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E}) \subset \mathbb{S}^q\mathrm{AP}(\mathbb{R},\mathbb{E}) \subset \mathbb{S}\mathrm{AP}(\mathbb{R},\mathbb{E}) \subset \mathbb{S}(\mathbb{R},\mathbb{E})$$

for $p \geq q \geq 1$ and $\mathrm{AP}(\mathbb{R},\mathbb{E}) = \mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E}) \cap \mathscr{C}_u(\mathbb{R},\mathbb{E})$, where $\mathscr{C}_u(\mathbb{R},\mathbb{E})$ denotes the set of $\mathbb{E}$-valued uniformly continuous functions on $\mathbb{R}$.

For more properties and details about real and Banach-valued Stepanov almost periodic functions, we refer the reader for instance to the papers and monographs [2,3,8,11,27, 43,51,52].

Beside the previous characterization of the class $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$, there is an other one based on the concept of Stepanov almost periodicity in Lebesgue measure, invented by Stepanov [73]. This concept plays a significant role in the proof of our superposition theorem in $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$ (Theorem 2.2.2).

Stepanov almost periodicity in Lebesgue measure For any measurable set $A \subset \mathbb{R}$, let

$$\varkappa(A) = \sup_{\xi\in\mathbb{R}} \operatorname{meas}([\xi,\xi+1]\cap A),$$

where meas is Lebesgue measure. A measurable function $f : \mathbb{R} \to \mathbb{E}$ is said to be *Stepanov almost periodic in Lebesgue measure* or *$\varkappa$-almost periodic* if for any ε, $\delta > 0$, the set

$$T_{\varkappa}(f,\varepsilon,\delta) := \left\{ \tau \in \mathbb{R}, \sup_{\xi\in\mathbb{R}} \text{meas}\left\{t \in [\xi,\xi+1], d\big(f(t+\tau), f(t)\big) \geq \varepsilon\right\} < \delta \right\}$$

is relatively dense. We denote by $\mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E})$ ($\mathscr{S}_{\varkappa}(\mathbb{R})$ when $\mathbb{E} = \mathbb{R}$) the space of such fonctions. This space was studied in depth by several authors (in both normed and metric spaces). One can mention Stoinski's works [74, 75], where an approximation property and some compactness criterion are given. Danilov [30, 31, 33] has explored this class in the framework of almost periodic measure-valued functions. The recently published paper [67], that we discovered at the time of writing this paper, completes the previous ones. The authors of this paper investigate some other properties, in particular, they show that in general the mean value of $\varkappa$-almost periodic functions may not exist, furthermore, $\varkappa$-almost periodic functions are generally not Stepanov-bounded.
As pointed out by Danilov [31], $\varkappa$-almost periodicity coincides with classical Stepanov almost periodicity when replacing the metric d by $d' = \min(d,1)$. In other words, we have the following characterization (see [31, 32])

$$\mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E}) = \mathbb{S}^1\text{AP}\big(\mathbb{R},(\mathbb{E},d')\big). \tag{1.18}$$

More generally, Stepanov almost periodicity can be seen as $\varkappa$-almost periodicity under a uniform integrability condition in Stepanov sense (see [73], [30]). To be more precise, let $M'_p(\mathbb{R},\mathbb{E})$ be the set of $D^d_{\mathbb{S}^p}$-bounded functions such that

$$\lim_{\delta\to 0^+} \sup_{\xi\in\mathbb{R}} \sup_{\substack{T\subset[\xi,\xi+1] \\ \text{meas}\, T\leq\delta}} \int_T d^p\big(f(t),x_0\big)dt = 0. \tag{1.19}$$

The space $M'_p(\mathbb{R},\mathbb{E})$, $p \geq 1$, is a closed subset of $(\mathbb{S}^p(\mathbb{R},\mathbb{E}), D^d_{\mathbb{S}^p})$. In [31, pp. 1420], Danilov gives an elegant characterization of Stepanov almost periodic functions in terms of $M'_p(\mathbb{R},\mathbb{E})$ and $\mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E})$, more precisely :

$$\mathbb{S}^p\text{AP}(\mathbb{R},\mathbb{E}) = \mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E}) \cap M'_p(\mathbb{R},\mathbb{E}). \tag{1.20}$$

A rather interesting result about the space $\mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E})$ is reported in the following theorem [32, Theorem 3], which gives a uniform approximation of Stepanov almost periodic functions by Bohr almost periodic functions, in the context of normed space $\mathbb{E}$. Before, let us denote by $\mathscr{S}(\mathbb{R})$ the collection of measurable sets $T \subset \mathbb{R}$ such that $\mathbf{1}_T \in \mathbb{S}\text{AP}(\mathbb{R})$, and by T^c the complementary set of T.

Theorem 1.2.1 (Danilov [32]). *Let $f \in \mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E})$, then for any $\delta > 0$, there exist a set $T_\delta \in \mathscr{S}(\mathbb{R})$ and a Bohr almost periodic function F_δ such that $\varkappa(T^c_\delta) < \delta$ and $f(t) = F_\delta(t)$ for all $t \in T_\delta$.*

As consequence, we have the following corollary :

Corollary 1.2.2. *Let $f \in \mathscr{S}_{\varkappa}(\mathbb{R},\mathbb{E})$. Then, for all $\varepsilon > 0$, there exist a measurable set $T_\varepsilon \in \mathscr{S}(\mathbb{R})$ and a compact subset K_ε of $\mathbb{E}$ such that $\varkappa(T^c_\delta) < \varepsilon$ and $f(t) \in K_\varepsilon, \forall t \in T_\varepsilon$.*

Danilov has shown that this property remains valid even in the metric framework [33].

Remark 1.2.3. *1. Unlike almost periodicity in Bohr sense and almost periodicity in Lebesque measure for function with values in a metric space* $(\mathbb{E}, d)$*, which depend only on the topological structure of* $\mathbb{E}$ *and not on its metric (see e.g., [10] and [31] respectively), Stepanov almost periodicity is a metric property. In fact, as the metrics* d *and* $d' = \min(d, 1)$ *are topologically equivalent on* $\mathbb{E}$*, we only need to show that the inclusion* $\mathbb{S}\mathrm{AP}\big(\mathbb{R}, (\mathbb{E}, d)\big) \subset \mathbb{S}\mathrm{AP}\big(\mathbb{R}, (\mathbb{E}, d')\big)$ *is strict, since in view of* (1.18)*, we have* $\mathbb{S}\mathrm{AP}\big(\mathbb{R}, (\mathbb{E}, d')\big) = \mathscr{S}_{\varkappa}(\mathbb{R}, \mathbb{E})$*. Take for instance the example given in [6, Remark 3.3]. As shown by the authors, the function* $g = \exp\left(\sum_{n=2}^{+\infty} g_n\right)$*, where* g_n *is the* $4n$*-periodic function, given by*

$$g_n(t) = \beta_n \left(1 - \frac{2}{\alpha_n} |x - n|\right) \chi_{[n - \frac{2}{\alpha_n}, n + \frac{2}{\alpha_n}]}, \, t \in [-2n, 2n],$$

with $\alpha_n = 1/n^5$ *and* $\beta_n = n^3$*, is not in* $\mathbb{S}\mathrm{AP}\big(\mathbb{R}\big)$*. Using Danilov's Corollary [31], we get that* g *belongs to* $\mathscr{S}_{\varkappa}(\mathbb{R})$*, as a superposition of a continuous function and a periodic, continuous and bounded function.*

2. *Still in the spirit of the link between the spaces* $\mathbb{S}\mathrm{AP}\big(\mathbb{R}\big)$ *and* $\mathscr{S}_{\varkappa}(\mathbb{R})$*, an interesting property established by Stoiński says that the inverse of any trigonometric polynomial with constant sign is* $\varkappa$*-almost periodic. In particular, the Levitan's function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(t) = \frac{1}{2+\cos(t)+\cos(2t)}$ *is* $\varkappa$*-almost periodic but not Stepanov almost periodic (see Example 4.2.1 and [67] for the second statement).*
3. *An immediate consequence of this statement is that the uniform integrability in Stepanov sense is a metric property, that is, the space* $M'_p(\mathbb{R}, (\mathbb{E}, d))$ *depends on the metric* d*.*

Let us now introduce Stepanov almost periodicity for functions a parameter.

Bohr and Stepanov almost periodic functions depending on a parameter

1. We say that a parametric function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ is *almost periodic with respect to the first variable, uniformly with respect to the second variable in bounded subsets of* $\mathbb{X}$ (respectively *in compact subsets of* $\mathbb{X}$) if, for every bounded (respectively compact) subset B of $\mathbb{X}$, the mapping $f : \mathbb{R} \to C(B, \mathbb{Y})$ is almost periodic. We denote by $\mathrm{APU}_b(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $\mathrm{APU}_c(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ respectively the spaces of such functions.
2. We say that a function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ is $\mathbb{S}^p$-almost periodic if, for every $x \in \mathbb{X}$, the $\mathbb{Y}$-valued function $f(., x)$ is $\mathbb{S}^p$-almost periodic. We denote by $\mathbb{S}^p\mathrm{AP}(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ the space of such functions.
3. If $f(., x)$ is $\mathbb{S}^p$-almost periodic uniformly with respect to $x \in K$ (resp. $x \in B$), for any compact (resp. bounded) subset K (resp. B) of $\mathbb{X}$, we say that f is $\mathbb{S}^p$-almost periodic uniformly with respect to the second variable in compact (resp. bounded) subset of $\mathbb{X}$. The space of such functions is denoted by $\mathbb{S}^p\mathrm{APU_c}(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ (resp. $\mathbb{S}^p\mathrm{APU_b}(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$).

Clearly, we have the following inclusions :

$$\mathbb{S}^p\mathrm{APU_b}(\mathbb{R} \times \mathbb{X}, \mathbb{Y}) \subset \mathbb{S}^p\mathrm{APU_c}(\mathbb{R} \times \mathbb{X}, \mathbb{Y}) \subset \mathbb{S}^p\mathrm{AP}(\mathbb{R} \times \mathbb{X}, \mathbb{Y}) \subset \mathfrak{L}^p(\mathbb{R} \times \mathbb{X}, \mathbb{Y}),$$

where $\mathfrak{L}^p(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ denotes the set of measurable functions $f:\mathbb{R}\times\mathbb{X}\to\mathbb{Y}$ such that, for all $x\in\mathbb{X}$; $f(.,x)\in\mathfrak{L}^p(\mathbb{R},\mathbb{Y})$.
The following proposition will be very useful in the sequel.

Proposition 1.2.4. *Let $\mathbb{Y}$ be a complete metric space, and let $\mathbb{X}$ be a complete separable metric space. Let $f\in\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ satisfying the following Lipschitz condition :*

$$\mathfrak{d}\left(f(t,x),f(t,y)\right)\le K(t)\mathfrak{d}\left(x,y\right), \tag{1.21}$$

where $K(.)$ is a positive function in $\mathbb{S}^p(\mathbb{R})$. Then for every real sequence $(\alpha_n^{'})$, there exist a subsequence $(\alpha_n)\subset(\alpha_n^{'})$ (independent of x) and a function $f^\infty\in\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ such that for every $t\in\mathbb{R}$ and $x\in\mathbb{X}$, we have

$$\lim_n\int_t^{t+1}\mathfrak{d}\left(f(s+\alpha_n,x),f^\infty(s,x)\right)^p ds=0. \tag{1.22}$$

Proof : Firstly, let us show that f^∞ is Lipschitz with respect to the second variable in the Stepanov metric sense. Let $x,y\in\mathbb{X}$. We consider a real sequence $(\alpha_n^{'})\subset\mathbb{R}$. Since $f\in\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$, for every $x\in\mathbb{X}$, we can find a subsequence $(\alpha_n)\subset(\alpha_n^{'})$ (depending on x) such that

$$\lim_n D^{\mathfrak{d}}_{\mathbb{S}_p}\left(f(.+\alpha_n,x)-f^\infty(.,x)\right)=0. \tag{1.23}$$

For the same reason, for every $y\in\mathbb{X}$, there exists a subsequence of (α_n), (depending on both x and y still noted (α_n) for simplicity) such that

$$\lim_n D^{\mathfrak{d}}_{\mathbb{S}_p}\left(f(.+\alpha_n,y),f^\infty(.,y)\right)=0. \tag{1.24}$$

Then, by (1.21), (1.23) and (1.24), we get

$$\begin{aligned} D^{\mathfrak{d}}_{\mathbb{S}_p}\left(f^\infty(.,x),f^\infty(.,y)\right) &\le \lim D^{\mathfrak{d}}_{\mathbb{S}_p}\left(f^\infty(.,x),f(.+\alpha_n,x)\right)+\lim D^{\mathfrak{d}}_{\mathbb{S}_p}\left(f(.+\alpha_n,x),f(.+\alpha_n,y)\right)\\ &\quad+\lim D^{\mathfrak{d}}_{\mathbb{S}_p}\left(f(.+\alpha_n,y),f^\infty(.,y)\right)\\ &\le \|K\|_{S^p}\mathfrak{d}(x,y). \end{aligned} \tag{1.25}$$

Secondly, let us show (1.22). Let $(\alpha_n^{'})$ be a real sequence. Since $\mathbb{X}$ is separable, let D be a dense countable subset of $\mathbb{X}$. Using (1.23) and a diagonal procedure, we can find a subsequence (α_n) of $(\alpha_n^{'})$ such that for every $t\in\mathbb{R}$ and $x\in D$, we have

$$\lim_n\int_t^{t+1}\left(\mathfrak{d}\left(f(s+\alpha_n,x),f^\infty(s,x)\right)\right)^p ds=0 \tag{1.26}$$

Let $x\in\mathbb{X}$, there exists a sequence $(x_k)\subset D$ such that $\lim\mathfrak{d}(x_k,x)=0$. From (1.21), we deduce

$$\lim_k\int_t^{t+1}\left(\mathfrak{d}\left(f(s+\alpha_n,x_k),f(s+\alpha_n,x)\right)\right)^p ds=0, \tag{1.27}$$

uniformly with respect to $n\in\mathbb{N}$. Now from (1.26), we obtain, for every $t\in\mathbb{R}$ and $k\in\mathbb{N}$,

$$\lim_n\int_t^{t+1}\left(\mathfrak{d}\left(f(s+\alpha_n,x_k),f^\infty(s,x_k)\right)\right)^p ds=0. \tag{1.28}$$

Using (1.27), (1.28) and by a classical result on interchange of limits, we deduce

$$\lim_n f(s+\alpha_n, x) = \lim_k f^\infty(s, x_k) = f^\infty(s, x)$$

in Stepanov metric. The last equality follows from (1.25). □

□

1.2.3 Bohr and Stepanov weighted pseudo almost periodic functions with values in metric space

The notions of Stepanov-like weighted pseudo almost periodicity and Stepanov pseudo almost periodicity of functions, with values in Banach space $\mathbb{X}$, were introduced by T. Diagana [34,35,36] as natural generalizations of the pseudo almost periodicity invented by Zhang [85, 86].
Here we give the definitions of these different notions for functions with values in a complete metric space $\mathbb{E}$. Constantin and Maria Tudor [78] have proposed an elegant definition of pseudo almost periodicity in the context of metric spaces, which is slightly restrictive, since it requires compactness of the range of the function instead of its boundedness. A more general definition of weighted pseudo almost periodicity has been introduced in [9], where it is shown that there is no need to assume that $\mathbb{E}$ is a vector space, nor a metric space, and these notions depend only on the topological structure of $\mathbb{E}$. The definition we propose here is intermediate to C. and M. Tudor's definition, and those in the wide sense [9, Proposition 2.5 (i)], and coincides with the one existing in the literature when $\mathbb{E}$ is a normed space. Let μ be a Borel measure on $\mathbb{R}$ satisfying 1.6

Definition 1.2.5. A continuous and bounded function $f : \mathbb{R} \to \mathbb{E}$ is said to be *pseudo almost periodic* if there exists a function $g \in \mathrm{AP}(\mathbb{R}, \mathbb{E})$ such that the mapping $t \to d(f(t), g(t))$ is in $\mathscr{E}(\mathbb{R}, \mathbb{R})$, and it's said to be *$\mu$-pseudo almost periodic* or *weighted pseudo almost periodic* if there exists a function $g \in \mathrm{AP}(\mathbb{R}, \mathbb{E})$ such that the mapping $t \to d(f(t), g(t))$ is in $\mathscr{E}(\mathbb{R}, \mathbb{R}, \mu)$.

We denote respectively by $\mathrm{PAP}(\mathbb{R}, \mathbb{E})$ and $\mathrm{PAP}(\mathbb{R}, \mathbb{E}, \mu)$ the spaces of such functions. Note that g is uniquely determined by f in the first case (see [78]). This is not necessarily the case when considering $f \in \mathrm{PAP}(\mathbb{R}, \mathbb{E}, \mu)$. However, it is easy to see that a sufficient condition of uniqueness of g is that $\mathscr{E}(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. This is the case in particular if Condition **(H)** of [16] is satisfied. Let $p \geq 1$. We use Bochner's transformation to define the Stepanov μ-pseudo almost periodic functions :

Definition 1.2.6. 1. We say that $f : \mathbb{R} \to \mathbb{E}$ is *$\mathbb{S}^p$-weighted pseudo almost periodic*, or *Stepanov μ-pseudo almost periodic* if

$$f^b \in \mathrm{PAP}(\mathbb{R}, \mathrm{L}^p(0,1;\mathbb{E}), \mu),$$

that is, if there exists $g \in \mathbb{S}^p\mathrm{AP}(\mathbb{R}, \mathbb{E})$ such that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \mathscr{D}_{\mathrm{L}^p}\Big(f^b(t), g^b(t)\Big)\, d\mu(t) = 0, \tag{1.29}$$

or, equivalently,

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_0^1 d^p(f(t+s),g(t+s))ds\right)^{1/p} d\mu(t)=0. \tag{1.30}$$

We denote by $\mathbb{S}^p\mathrm{PAP}(\mathbb{R},\mathbb{E},\mu)$ the space of such functions. Note that, in this case, the function g is uniquely determined if μ satisfies Condition (**H**). In fact, assume that $g_1,g_2\in\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$ define the same function f. Then, the mapping $t\to\mathscr{D}_{\mathrm{L}^p}(g_1^b(t),g_2^b(t))$ is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)\cap\mathrm{AP}(\mathbb{R},\mathbb{R})$. It follows that $\mathscr{D}_{\mathrm{L}^p}(g_1^b(t),g_2^b(t))=0$, for all $t\in\mathbb{R}$. Consequently $g_1=g_2$, *a.e.*.
We have the following characterization of $\mathbb{S}^p\mathrm{PAP}(\mathbb{R},\mathbb{E},\mu)$:

Proposition 1.2.7. *Eq.* (1.30) *is equivalent to*

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\int_0^1 d^p(f(t+s),g(t+s))ds\,d\mu(t)=0. \tag{1.31}$$

Our proof is inspired from J. Blot and P. Cieutat [15, Proposition 6.6].

Proof : The case when $p=1$ is obvious. Set, for simplicity, $d(f(t+s),g(t+s)):=h(t+s)$ and $|H|_{\mathbb{S}^p(t)}:=\left(\int_0^1\|H(t+s)\|^p ds\right)^{1/p}$, for any p-locally integrable Banach-space valued function, H. Let us assume that $t\mapsto|h|^p_{\mathbb{S}^p(t)}\in\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$, $p>1$. Using Hölder's inequality, we have

$$\int_{[-r,r]}|h|_{\mathbb{S}^p(t)}\,d\mu(t)\le(\mu([-r,r]))^{1/q}\left\{\int_{[-r,r]}|h|^p_{\mathbb{S}^p(t)}\,d\mu(t)\right\}^{1/p}.$$

Thus,

$$\frac{1}{\mu([-r,r])}\int_{[-r,r]}|h|_{\mathbb{S}^p(t)}\,d\mu(t)\le\left\{\frac{1}{\mu([-r,r])}\int_{[-r,r]}|h|^p_{\mathbb{S}^p(t)}\,d\mu(t)\right\}^{1/p}$$

which leads to $|h|_{\mathbb{S}^p(.)}\in\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$. Conversely, set $M=\sup_{t\in\mathbb{R}}|h|^{p-1}_{\mathbb{S}^p(t)}<\infty$. We have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{[-r,r]}|h|^p_{\mathbb{S}^p(t)}\,d\mu(t)\le&\frac{1}{\mu([-r,r])}\int_{[-r,r]}|h|_{\mathbb{S}^p(t)}|h|^{p-1}_{\mathbb{S}^p(t)}\,d\mu(t)\\ \le&M\frac{1}{\mu([-r,r])}\int_{[-r,r]}|h|_{\mathbb{S}^p(t)}\,d\mu(t)\end{aligned}$$

which means that $|h|^p_{\mathbb{S}^p(.)}\in\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$.

□

From now on, we only deal with the weighted pseudo almost periodicity since pseudo almost periodicity is a special case.

Weighted pseudo almost periodic and Stepanov-like weighted pseudo almost periodic functions depending on a parameter Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6). The definition of μ-pseudo almost periodicity for functions with parameter is the same as in [9].

– We say that a continuous and bounded function $f : \mathbb{R}\times\mathbb{X}\to\mathbb{Y}$ is *μ-pseudo almost periodic with respect to the first variable, uniformly with respect to the second variable in compact subsets of* $\mathbb{X}$ if, for every $x\in\mathbb{X}$, $f(.,x)$ is μ-pseudo almost periodic (in this case, we write $f\in \mathrm{PAP}(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$, and there exists $g\in \mathrm{APU}_c(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ such that the convergence of $1/\mu([-r,r])\int_{-r}^{r} d(f(t,x),g(t,x))\,d\mu(t)$ is uniform with respect to x in compact subsets of $\mathbb{X}$. The space of such functions is denoted by $\mathrm{PAPU}_c(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.
– If $(\mathbb{Y};\|.\|)$ is a Banach space, we say that a function $f\in\mathfrak{L}^p(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is *Stepanov-like μ-ergodic with respect to the first variable, uniformly with respect to the second variable in compact subsets of* $\mathbb{X}$) if, for every $x\in\mathbb{X}$, $f^b(.,x)$ is μ-ergodic, and the convergence of
$$1/\mu([-r,r])\int_{-r}^{r}|f(.,x)|_{\mathbb{S}^p(t)}\,d\mu(t)$$
is uniform with respect to x in compact subsets of $\mathbb{X}$. The space of such functions is denoted by $\mathbb{S}^p\mathscr{E}\mathrm{U}_c(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.
– A function $f\in\mathfrak{L}^p(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ is said to be *Stepanov-like μ-pseudo almost periodic with respect to the first variable, uniformly with respect to the second variable in compact subsets of* $\mathbb{X}$ if there exists $g\in\mathbb{S}^p\mathrm{APU}_c(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ such that
$$\Big[(t,x)\mapsto d\Big(f(t,x),g(t,x)\Big)\Big]\in\mathbb{S}^p\mathscr{E}\mathrm{U}_c(\mathbb{R}\times\mathbb{X},\mathbb{R},\mu).$$
The space of such functions is denoted by $\mathbb{S}^p\mathrm{PAPU}_c(\mathbb{R}\times\mathbb{X},\mathbb{Y},\mu)$.

1.2.4 Weighted pseudo almost periodicity for stochastic processes

To define weighted pseudo almost periodicity in pth mean, we assume that $(\mathbb{X},\|.\|)$ is a separable Banach space.

Weighted pseudo almost periodicity in pth mean

Let $X=(X_t)_{t\in\mathbb{R}}$ be a continuous stochastic process with values in $\mathbb{X}$, defined on a probability space $(\Omega,\mathscr{F},\mathrm{P})$. Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6).
Let $p>0$. We say that X is *almost periodic in pth mean* (respectively *μ-pseudo almost periodic in pth mean*) if the mapping $t\mapsto X(t)$ is in $\mathrm{AP}(\mathbb{R},\mathrm{L}^p(\Omega,\mathrm{P},\mathbb{X}))$ (respectively in $\mathrm{PAP}(\mathbb{R},\mathrm{L}^p(\Omega,\mathrm{P},\mathbb{X}),\mu)$, i.e., if it has the form $X=Y+Z$, where $Y\in\mathrm{AP}(\mathbb{R},\mathrm{L}^p(\Omega,\mathrm{P},\mathbb{X}))$ and $Z\in\mathscr{E}(\mathbb{R},\mathrm{L}^p(\Omega,\mathrm{P},\mathbb{X}),\mu)$). When $p=2$, we say that X is *square-mean almost periodic* (respectively *square-mean μ-pseudo almost periodic*).
The process X is said to be *almost periodic in probability* if the mapping $X:t\to\mathrm{L}^0(\Omega,\mathrm{P},\mathbb{X})$ is almost periodic.
The process X is said to be *μ-pseudo almost periodic in probability*, and we write $X\in\mathrm{PAP}(\mathbb{R},\mathrm{L}^0(\Omega,\mathrm{P},\mathbb{X}),\mu)$, if the mapping $t\mapsto X(t)$, $\mathbb{R}\to\mathrm{L}^0(\Omega,\mathrm{P},\mathbb{X})$ is μ-pseudo almost per-

iodic, i.e. if it has the form $X = Y + Z$ where $Y \in \mathrm{AP}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X}))$ and Z satisfies

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \mathrm{E}\left(\|Z(t)\| \wedge 1 \right) d\mu(t) = 0. \tag{1.32}$$

We denote by $\mathscr{E}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X}), \mu)$ the set of stochastic processes Z satisfying (1.32).

Weighted pseudo almost periodicity in p-distribution

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Polish spaces. In what follows, we recall a useful variant of (weighted pseudo) almost periodic in distribution, that combines the well-known almost periodicity in distribution [77] and a uniform integrability condition.
Let $\tau : \mathbb{X} \to \mathbb{Y}$ a Borel measurable mapping and μ is a Borel measure on $\mathbb{X}$.
If X is continuous with values in $\mathbb{X}$, we denote by $\widetilde{X}(t)$ the random variable $X(t + .)$ with values in $\mathrm{C}(\mathbb{R}, \mathbb{X})$.
Following Tudor's terminology [77], we say that X is *almost periodic in one-dimensional distributions* if the mapping $t \mapsto \mathrm{law}(X(t))$, $\mathbb{R} \mapsto \mathscr{M}^{1,+}(\mathbb{X})$ is almost periodic.
If X has continuous trajectories, we say that X is *almost periodic in distribution*, and write $X \in \mathrm{APD}(\mathbb{R}, \mathbb{X})$, if the mapping $t \mapsto \mathrm{law}(\widetilde{X}(t))$, $\mathbb{R} \mapsto \mathscr{M}^{1,+}(\mathrm{C}_k(\mathbb{R}, \mathbb{X}))$ is almost periodic.
Let us recall the definition of almost periodicity in p-distribution that we introduced in [9], which takes into account integrability of order p.

Definition 1.2.8 ([9]). Let $p \geq 0$. A continuous $\mathbb{X}$-valued stochastic process is called *almost periodic in p-distribution* if

(i) $X \in \mathrm{APD}(\mathbb{R}, \mathbb{X})$,

(ii) if $p > 0$, the family $(\|X(t)\|^p)_{t\in\mathbb{R}}$ is uniformly integrable.

We denote by $\mathrm{APD}^p(\mathbb{R}, \mathbb{X})$ the set of $\mathbb{X}$-valued processes which are almost periodic in p-distribution, in particular we have $\mathrm{APD}^0(\mathbb{R}, \mathbb{X}) = \mathrm{APD}(\mathbb{R}, \mathbb{X})$.
The previous conditions imply that the mapping $t \mapsto X(t)$, $\mathbb{R} \to \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X})$, is continuous. In the same way, for $p \geq 0$, one defines the sets $\mathrm{APD}^p_f(\mathbb{R}, \mathbb{X})$ and $\mathrm{APD}^p_1(\mathbb{R}, \mathbb{X})$ of processes which are respectively *almost periodic in one-dimensional p-distributions* and *almost periodic in finite dimensional p-distributions*.
The following definition proposed in [9] turns out useful in the context of stochastic differential equation.

Definition 1.2.9 ([9]). Assume that $\mathbb{X}$ is a vector space. Let $p \geq 0$. We say that X is *μ-pseudo almost periodic in p-distribution* if X can be written

$$X = Y + Z, \text{ where } Y \in \mathrm{APD}^p(\mathbb{R}, \mathbb{X}) \text{ and } Z \in \mathscr{E}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu).$$

The set of $\mathbb{X}$-valued processes which are μ-pseudo almost periodic in p-distribution is denoted by $\mathrm{PAPD}^p(\mathbb{R}, \mathbb{X})$. Similar definitions hold for the spaces $\mathrm{PAPD}^p_1(\mathbb{R}, \mathbb{X}, \mu)$ and $\mathrm{PAPD}^p_f(\mathbb{R}, \mathbb{X}, \mu)$ of processes which are *μ-pseudo almost periodic in one-dimensional p-distributions* and *in finite dimensional p-distributions* respectively.

CHAPITRE

2

Some superposition theorems

2.1 Superposition theorems for almost automorphy type functions and processes

Almost automorphy in distribution and some of its variants enjoy some stability properties, as shows the following superposition lemma. Similar results could be proved by the same method for one-dimensional or for finite dimensional distributions.

Theorem 2.1.1. *(Superposition lemma) Let X be a continuous $\mathbb{X}$-valued stochastic process, and let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{Y}$ be a continuous mapping.*

1. *If X is almost periodic in distribution and f is almost periodic with respect to the first variable, uniformly with respect to the second variable in compact subsets of $\mathbb{X}$, then $f(.,X(.))$ is almost periodic in distribution.*
2. *If X is almost automorphic in distribution and f is compact almost automorphic with respect to the first variable, uniformly with respect to the second variable in compact subsets of $\mathbb{X}$, then $f(.,X(.))$ is almost automorphic in distribution.*
3. *Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6), and let $p \geq 0$. Assume that f is μ-pseudo compact almost automorphic with respect to the first variable, uniformly with respect to the second variable in compact subsets of $\mathbb{X}$, i.e.*

   $$f = g + h, \text{ with } g \in AA_cU_c(\mathbb{R} \times \mathbb{X}, \mathbb{Y}) \text{ and } h \in \mathscr{E}U_c(\mathbb{R} \times \mathbb{X}, \mathbb{Y}, \mu).$$

   Assume that g is continuous with respect to the second variable. Assume furthermore that f is uniformly continuous in the second variable in compact subsets of $\mathbb{X}$, uniformly with respect to the first variable, that is,

for every $\varepsilon > 0$, and for every compact subset K of $\mathbb{X}$,
there exists $\eta > 0$ such that, for all $x, y \in K$,

$$\|x - y\| \leq \eta \Rightarrow \sup_{t \in \mathbb{R}} \|f(t,x) - f(t,y)\| \leq \varepsilon. \quad (2.1)$$

Let X be μ-pseudo almost automorphic in p-distribution, with decomposition (Y,Z), namely,

$$X = Y + Z, Y \in AAD^p(\mathbb{R},\mathbb{X}), Z \in \mathscr{E}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X}), \mu).$$

If $p > 0$, assume also that f and g satisfy the growth condition

$$\|f(t,x)\| + \|g(t,x)\| \leq C(1 + \|x\|) \quad (2.2)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{X}$ and for some constant C, and that, either the family $(\|Z(t)\|^p)_{t\in\mathbb{R}}$ is uniformly integrable, or f is Lipschitz with respect to the second variable, uniformly with respect to the first one. Then $f(.,X(.))$ is μ-pseudo almost automorphic in p-distribution.

Proof : We only prove the second and third items, the first one can be proved in the same way as 2, using, for example, Bochner's double sequence criterion.

2. For each $t \in \mathbb{R}$, for each $x \in \mathrm{C}_k(\mathbb{R},\mathbb{X})$, and for each $s \in \mathbb{R}$, let us denote

$$\widetilde{f}(t,x)(s) = f(s+t, x(s)).$$

By continuity of f on $\mathbb{R} \times \mathbb{X}$, $\widetilde{f}(t,.)$ maps $\mathrm{C}_k(\mathbb{R},\mathbb{X})$ to $\mathrm{C}_k(\mathbb{R},\mathbb{Y})$.
Let $\mathscr{K}$ be a compact subset of $\mathrm{C}_k(\mathbb{R},\mathbb{X})$. By the Arzelà-Ascoli Theorem (see e.g. [41, Theorems 8.2.10 and 8.2.11]), this means that $\mathscr{K}$ is closed in $\mathrm{C}_k(\mathbb{R},\mathbb{X})$ and equicontinuous, and that, for every compact interval I of $\mathbb{R}$, the set $\{x(t); x \in \mathscr{K},\ t \in I\}$ has compact closure in $\mathbb{X}$. Let $t \in \mathbb{R}$, and let (t_n) be a sequence in $\mathbb{R}$ converging to t. Let I be a compact interval of $\mathbb{R}$, and let K be the closure of $\{x(s); x \in \mathscr{K},\ s \in I\}$. We have, for any $y \in K$, and for any $s \in \mathbb{R}$,

$$\lim_n f(t_n + s, y) = f(t + s, y)$$

where the convergence is uniform with respect to $y \in K$ and $s \in I$, because f is compact almost automorphic uniformly with respect to the second variable in compact subsets of $\mathbb{X}$. In particular we have, uniformly with respect to $x \in \mathscr{K}$ and $s \in I$,

$$\lim_n \widetilde{f}(t_n, x)(s) = \lim_n f(t_n + s, x(s)) = f(t + s, x(s)) = \widetilde{f}(t,x)(s),$$

which proves that the mapping $\widetilde{f}(.,x) : \mathbb{R} \to \mathrm{C}_k(\mathbb{R},\mathbb{X})$, is continuous, uniformly with respect to x in compact subsets of $\mathrm{C}_k(\mathbb{R},\mathbb{X})$.
Let us check that $\widetilde{f} : \mathbb{R} \times \mathrm{C}_k(\mathbb{R},\mathbb{X}) \to \mathrm{C}_k(\mathbb{R},\mathbb{X})$ is compact almost automorphic with respect to the first variable, uniformly with respect to the second variable in compact subsets of $\mathrm{C}_k(\mathbb{R},\mathbb{X})$. Let (t'_n) be a sequence in $\mathbb{R}$. There exists a subsequence (t_n) such that, for every $t \in \mathbb{R}$ and every $y \in \mathbb{X}$,

$$\lim_n \lim_m f(t + t_n - t_m, y) = f(t,y),$$

where the convergence is uniform with respect to y in compact subsets of $\mathbb{X}$ and t in compact intervals of $\mathbb{R}$. For each $t \in \mathbb{R}$, and for each $s \in \mathbb{R}$, we have

$$\begin{aligned}\lim_n \lim_m \widetilde{f}(t+t_n-t_m,x)(s) &= \lim_n \lim_m f(s+t+t_n-t_m,x(s)) \\ &= f(s+t,x(s)) = \widetilde{f}(t,x)(s),\end{aligned}$$

and these convergences are uniform with respect to t and s in compact intervals, and with respect to $x \in \mathscr{K}$, which proves our claim.
Let $\varphi : \mathrm{C}_k(\mathbb{R},\mathbb{Y}) \to \mathbb{R}$ be bounded Lipschitz, with $\|\varphi\|_{\mathrm{BL}} \le 1$, where $\|.\|_{\mathrm{BL}}$ is taken relatively to a distance $\underline{\mathfrak{d}}$ which generates the topology of $\mathrm{C}_k(\mathbb{R},\mathbb{Y})$, for example the distance defined by (1.14). Let (t'_n) be a sequence in $\mathbb{R}$. Let (t_n) be a subsequence such that, for every $t \in \mathbb{R}$ and every $y \in \mathbb{X}$,

$$\begin{aligned}\lim_n \lim_m f(t+t_n-t_m,y) &= f(t,y),\\ \lim_n \lim_m \mathrm{law}(\widetilde{X}(t+t_n-t_m)) &= \mathrm{law}(\widetilde{X}(t)).\end{aligned}$$

Let $\varepsilon > 0$. We can find a compact subset $\mathscr{K}_\varepsilon$ of $\mathrm{C}_k(\mathbb{R},\mathbb{X})$ such that, for every $t \in \mathbb{R}$,

$$\mathrm{P}\left\{\widetilde{X}(t) \in \mathscr{K}_\varepsilon\right\} \ge 1-\varepsilon.$$

Let $t \in \mathbb{R}$ be fixed, and, for all n,m, let $\Omega_{\varepsilon,n,m}$ be the measurable subset of Ω on which $\widetilde{X}(t+t_n-t_m) \in \mathscr{K}_\varepsilon$. We have

$$\begin{aligned}&\left|\mathrm{E}\left(\varphi\circ\widetilde{f}(t+t_n-t_m,\widetilde{X}(t+t_n-t_m)) - \varphi\circ\widetilde{f}(t,\widetilde{X}(t))\right)\right|\\
&\quad\le \left|\mathrm{E}\left(\varphi\circ\widetilde{f}(t+t_n-t_m,\widetilde{X}(t+t_n-t_m)) - \varphi\circ\widetilde{f}(t,\widetilde{X}(t+t_n-t_m))\right)\right|\\
&\qquad + \left|\mathrm{E}\left(\varphi\circ\widetilde{f}(t,\widetilde{X}(t+t_n-t_m)) - \varphi\circ\widetilde{f}(t,\widetilde{X}(t))\right)\right|\\
&\quad\le \mathrm{E}\left(\mathbf{1}_{\Omega_{\varepsilon,n,m}}\underline{\mathfrak{d}}\Big(\widetilde{f}(t+t_n-t_m,\widetilde{X}(t+t_n-t_m)),\widetilde{f}(t,\widetilde{X}(t+t_n-t_m))\Big)\right)\\
&\qquad + \mathrm{E}\left(\mathbf{1}_{\Omega^c_{\varepsilon,n,m}}\left|\varphi\circ\widetilde{f}(t+t_n-t_m,\widetilde{X}(t+t_n-t_m)) - \varphi\circ\widetilde{f}(t,\widetilde{X}(t+t_n-t_m)))\right|\right)\\
&\qquad + \left|\mathrm{E}\left(\varphi\circ\widetilde{f}(t,\widetilde{X}(t+t_n-t_m)) - \varphi\circ\widetilde{f}(t,\widetilde{X}(t))\right)\right|\\
&\quad= A_{n,m} + B_{n,m} + C_{n,m}.\end{aligned}$$

We have

$$A_{n,m} \le \mathrm{E}\left(\mathbf{1}_{\Omega_\varepsilon}\sup_{x\in\mathscr{K}_\varepsilon}\underline{\mathfrak{d}}\Big(\widetilde{f}(t+t_n-t_m,x),\widetilde{f}(t,x)\Big)\right) \le \sup_{x\in\mathscr{K}_\varepsilon}\underline{\mathfrak{d}}\Big(\widetilde{f}(t+t_n-t_m,x),\widetilde{f}(t,x)\Big)$$

thus by the almost automorphy property of $\widetilde{f}$, we have $\lim_n\lim_m A_{n,m} = 0$. Furthermore, $B_{n,m} \le 2\mathrm{P}(\Omega^c_{\varepsilon,n,m}) \le 2\varepsilon$ because $\|\varphi\|_{\mathrm{BL}} \le 1$. Finally, $\lim_n\lim_m C_{n,m} = 0$ by boundedness and continuity of $\widetilde{f}(t,.) : \mathrm{C}_k(\mathbb{R},\mathbb{X}) \to \mathrm{C}_k(\mathbb{R},\mathbb{X})$ and the convergence in distribution of $\widetilde{X}(t+t_n-t_m)$ to $\widetilde{X}(t)$ for each $t \in \mathbb{R}$. As ε and φ are arbitrary, we have proved that

$$\lim_n\lim_m \mathrm{law}(\widetilde{f}(t+t_n-t_m,\widetilde{X}(t+t_n-t_m))) = \mathrm{law}(\widetilde{f}(t,\widetilde{X}(t))),$$

thus the mapping $t \mapsto \text{law}(\widetilde{f}(t, \widetilde{X}(t)))$ is almost automorphic in distribution.

3. We use ideas of the proof of [16, Theorem 5.7]. The function $f(.,X(.))$ can be decomposed as

$$f(t,X(t)) = g(t,Y(t)) + f(t,X(t)) - f(t,Y(t)) + h(t,Y(t)).$$

Let $G(t) = g(t,Y(t))$ and $H(t) = f(t,X(t)) - f(t,Y(t)) + h(t,Y(t))$. By using 2. and (2.2), we see that $t \mapsto G(t)$ is in $\text{AAD}^p(\mathbb{R},\mathbb{Y})$. Furthermore, by (2.2), the continuity of f and the μ-pseudo almost automorphy in p-distribution of X, we have, using Vitali's theorem, that $f(.,X(.))$ is a continuous $\text{L}^p(\Omega,\text{P},\mathbb{Y})$-valued function. Indeed, if $t_n \to t$, then the sequence $(\|X(t_n)\|^p)$ is uniformly integrable, by continuity of the mapping $t \mapsto X(t)$, $\mathbb{R} \to \text{L}^p(\Omega,\text{P},\mathbb{X})$, and this entails that $(f(t_n,X(t_n)))$ is uniformly integrable. To show that $f(.,X(.))$ is in $\text{PAAD}^p(\mathbb{R},\mathbb{Y})$, it is enough to prove that $H \in \mathscr{E}(\mathbb{R},\text{L}^p(\Omega,\text{P},\mathbb{Y}),\mu)$.
Clearly H is in $\text{C}(\mathbb{R},\text{L}^p(\Omega,\text{P},\mathbb{Y}))$, and bounded in $\text{L}^p(\Omega,\text{P},\mathbb{X})$ (when $p > 0$), by (2.2). As Y is in $\text{AAD}(\mathbb{R},\mathbb{X})$, the family $(\widetilde{Y}(t))_{t\in\mathbb{R}} = (Y(t+.))_{t\in\mathbb{R}}$ is uniformly tight in $\text{C}_k(\mathbb{R},\mathbb{X})$. For each $\varepsilon > 0$, there exists a compact subset $\mathscr{K}_\varepsilon$ of $\text{C}_k(\mathbb{R},\mathbb{X})$ such that, for every $t \in \mathbb{R}$,

$$\text{P}\left\{\widetilde{Y}(t) \in \mathscr{K}_\varepsilon\right\} \geq 1-\varepsilon.$$

By the Arzelà-Ascoli Theorem (see e.g. [41, Theorems 8.2.10 and 8.2.11]), this implies that, for every $\varepsilon > 0$, and for every compact interval I of $\mathbb{R}$, there exists a compact subset $K_{\varepsilon,I}$ such that, for every $t \in \mathbb{R}$;

$$\text{P}\{(\forall s \in I)\ Y(t+s) \in K_{\varepsilon,I}\} \geq 1-\varepsilon.$$

In particular, we have, for each integer n,

$$\text{P}\left\{(\forall t \in [n,n+1])\ Y(t) \in K_{\varepsilon,[0,1]}\right\} \geq 1-\varepsilon.$$

Let $\Omega_{\varepsilon,n}$ be the measurable subset of Ω on which $Y(t) \in K_{\varepsilon,[0,1]}$ for all $t \in [n,n+1]$. The function g is uniformly continuous on $\mathbb{R} \times K_{\varepsilon,[0,1]}$ by Proposition 1.1.1. We deduce by (2.1) that there exists $\eta(\varepsilon) > 0$ such that, for all $y_1,y_2 \in K_{\varepsilon,[0,1]}$,

$$\|y_1 - y_2\| \leq \eta(\varepsilon) \Rightarrow \sup_{t\in\mathbb{R}}\big(\|h(t,y_1) - h(t,y_2)\|\big) \leq \varepsilon.$$

We can find a finite sequence $(y_i)_{1\leq i\leq m}$ in $K_{\varepsilon,[0,1]}$ such that

$$K_{\varepsilon,[0,1]} \subset \bigcup_{i=1}^m B(y_i,\eta(\varepsilon)).$$

Let $t \in \mathbb{R}$, and let n be an integer such that $t \in [n,n+1]$. We have

$$\begin{aligned}
&\text{E}\big(\|h(t,Y(t))\| \wedge 1\big)\\
&\qquad\leq \text{E}\left(\min_{1\leq i\leq m}\big(\mathbf{1}_{\Omega_{\varepsilon,n}}\|h(t,Y(t)) - h(t,y_i)\| \wedge 1\big)\right) + \max_{1\leq i\leq m}\|h(t,y_i)\|\\
&\qquad\quad + \text{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,n}}\|h(t,Y(t))\| \wedge 1\big)\\
&\qquad\leq \varepsilon + \max_{1\leq i\leq m}\|h(t,y_i)\| \wedge 1 + \text{P}\left(\Omega^c_{\varepsilon,n}\right)\\
&\qquad\leq \max_{1\leq i\leq m}\|h(t,y_i)\| + 2\varepsilon.
\end{aligned}$$

Since for all $i \in \{1,\dots,m\}$, the function $t \to h(t,y_i)$ satisfies

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|h(t,y_i)\| \, d\mu(t) = 0,$$

we deduce that, for every $\varepsilon > 0$,

$$\limsup_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \mathrm{E}\big(\|h(t,Y(t))\| \wedge 1\big) d\mu(t) \le 2\varepsilon.$$

This shows that $t \to h(t,Y(t))$ is in $\mathscr{E}(\mathbb{R},\mathrm{L}^0(\Omega,\mathrm{P},\mathbb{Y}),\mu)$.
For $p > 0$, let $\delta > 0$. From the uniform integrability of $\big(\|h(t,Y(t))\|^p\big)_{t\in\mathbb{R}}$ (thanks to (2.2) and the uniform integrability of $(\|Y(t)\|^p)_{t\in\mathbb{R}}$), we can choose ε small enough such that, for any measurable $A \subset \Omega$ such that $\mathrm{P}(A) < \varepsilon$,

$$\sup_{t\in\mathbb{R}} \mathrm{E}\,(\,\mathbf{1}_A \|h(t,Y(t)\|^p) < \delta.$$

Note also that, for $p < 1$, the mapping $U \mapsto (\mathrm{E}\,\|U\|^p)^{1/p}$, $\mathrm{L}^p \to \mathbb{R}$, does not satisfy the triangular inequality. However, the mapping $(U_1,U_2) \mapsto \mathrm{E}\,\|U_1 - U_2\|^p$ is a distance on L^p. We deduce that, for all $U_1,U_2,U_3 \in \mathrm{L}^p$,

$$(\mathrm{E}\,\|U_1+U_2+U_3\|^p)^{1/p} \le 3^{1/p-1} \Big((\mathrm{E}\,\|U_1\|^p)^{1/p} + (\mathrm{E}\,\|U_2\|^p)^{1/p} + (\mathrm{E}\,\|U_3\|^p)^{1/p} \Big) .$$

To cover simultaneously the cases $p < 1$ and $p \ge 1$, we set $\kappa = \max(1, 3^{1/p-1})$. Using the same method as in the case when $p = 0$, we get

$$\begin{aligned}
\big(\mathrm{E}\,\|h(t,Y(t))\|^p\big)^{1/p} & \\
&\le \kappa \bigg(\mathrm{E}\Big(\min_{1\le i\le m} \mathbf{1}_{\Omega_{\varepsilon,n}} \|h(t,Y(t)) - h(t,y_i)\|^p \Big) \bigg)^{1/p} + \kappa \max_{1\le i\le m} \|h(t,y_i)\| \\
&\quad + \kappa \bigg(\mathrm{E}\Big(\mathbf{1}_{\Omega^c_{\varepsilon,n}} \|h(t,Y(t))\|^p \Big) \bigg)^{1/p} \\
&\le \kappa \Big(\max_{1\le i\le m} \|h(t,y_i)\| + \varepsilon + \delta \Big).
\end{aligned}$$

We conclude, using the ergodicity of $h(t,y_i)$ for all $i \in \{1,\dots,m\}$, that $t \to h(t,Y(t))$ is in $\mathscr{E}(\mathbb{R},\mathrm{L}^p(\Omega,\mathrm{P},\mathbb{Y}),\mu)$.
Secondly, we show that $F(.) := f(.,X(.)) - f(.,Y(.))$ is in $\mathscr{E}(\mathbb{R},\mathrm{L}^0(\Omega,\mathrm{P},\mathbb{Y}),\mu)$. Let

$$\Phi : \begin{cases} \mathbb{X} & \to \quad \mathrm{C}(\mathbb{R},\mathbb{Y}) \\ x & \mapsto \quad f(.,x). \end{cases}$$

Let us endow $\mathrm{C}(\mathbb{R},\mathbb{Y})$ with the distance $\mathfrak{d}_\infty(\varphi,\psi) = \sup_{t\in\mathbb{R}} \|\varphi(t) - \psi(t)\|$. By the uniform continuity assumption, Φ is continuous. By [16, Lemma 5.6], for each $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $t \in \mathbb{R}$, and for all $x,y \in \mathbb{X}$,

$$\Big(x \in K_{\varepsilon,[0,1]} \text{ and } \mathfrak{d}(x,y) \le \eta \Big) \;\Rightarrow \mathfrak{d}_\infty\Big(\Phi(x),\Phi(y)\Big) \le \varepsilon. \tag{2.3}$$

For each $t \in \mathbb{R}$, let $\Omega_{\varepsilon,t}$ be the subset of Ω on which $Y(t) \in K_{\varepsilon,[0,1]}$. Since $Z(t) = X(t) - Y(t)$, we obtain, the following inequalities, with the help of (2.3) and Chebyshev's inequality :

$$\begin{aligned}
&\frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\|F(t)\| \wedge 1\big) > 3\varepsilon\right\}}{\mu([-r,r])}\\
&\quad\le \frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\mathbf{1}_{\Omega_{\varepsilon,t}} \mathbf{1}_{\{\|Z(t)\|>\eta\}}(\|F(t)\| \wedge 1)\big) > \varepsilon\right\}}{\mu([-r,r])}\\
&\qquad + \frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\mathbf{1}_{\Omega_{\varepsilon,t}} \mathbf{1}_{\{\|Z(t)\|\le\eta\}}(\|F(t)\| \wedge 1)\big) > \varepsilon\right\}}{\mu([-r,r])}\\
&\qquad + \frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,t}}(\|F(t)\| \wedge 1)\big) > \varepsilon\right\}}{\mu([-r,r])}\\
&\quad= \frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\mathbf{1}_{\Omega_{\varepsilon,t}} \mathbf{1}_{\{\|Z(t)\|>\eta\}}(\|F(t)\| \wedge 1)\big) > \varepsilon\right\}}{\mu([-r,r])}\\
&\qquad + \frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,t}}(\|F(t)\| \wedge 1)\big) > \varepsilon\right\}}{\mu([-r,r])}\\
&\quad\le \frac{\mu\left\{t \in [-r,r]; \mathrm{P}\{\|Z(t)\| > \eta\} > \varepsilon\right\}}{\mu([-r,r])}\\
&\qquad + \frac{\mu\left\{t \in [-r,r]; \mathrm{P}(\Omega^c_{\varepsilon,t}) > \varepsilon\right\}}{\mu([-r,r])}\\
&\quad\le \frac{\mu\left\{t \in [-r,r]; \frac{1}{\eta}\mathrm{E}\big(\|Z(t)\|\big) > \varepsilon\right\}}{\mu([-r,r])}.
\end{aligned}$$

Since Z is in $\mathscr{E}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X}), \mu)$, we have, for the above ε,

$$\lim_{r\to\infty} \frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\|Z(t)\|\big) > \varepsilon\eta\right\}}{\mu([-r,r])} = 0,$$

which implies, using [16, Theorem 2.14] (see Remark 1.1.4),

$$\limsup_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \mathrm{E}\big(\|f(t,X(t)) - f(t,Y(t))\| \wedge 1\big) d\mu(t) = 0.$$

Therefore $t \to f(t,X(t)) - f(t,Y(t))$ is in $\mathscr{E}(\mathbb{R}, \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{Y}), \mu)$.
Assume now that $p > 0$. If f is Lipschitz with respect to the second variable, uniformly with respect to the first one, then $\|F(t)\| \le K\|Z(t)\|$ for some constant K, thus, trivially, $F \in \mathscr{E}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{Y}), \mu)$. If the family $(\|Z(t)\|^p)_{t\in\mathbb{R}}$ is uniformly integrable, then $(\|F(t)\|^p)_{t\in\mathbb{R}}$ is also uniformly integrable. Then, we can use the same reasoning as for $p = 0$, replacing $\|F(t)\| \wedge 1$ by $\|F(t)\|^p$, since $\mathrm{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,t}}(\|F(t)\|^p)\big)$ can be made arbitrarily small for ε sufficiently small. We obtain

$$\frac{\mu\left\{t \in [-r,r]; \mathrm{E}\big(\|F(t)\|^p\big) > 3\varepsilon\right\}}{\mu([-r,r])} \le \frac{\mu\left\{t \in [-r,r]; \frac{1}{\eta^p}\mathrm{E}\big(\|Z(t)\|^p\big) > \varepsilon\right\}}{\mu([-r,r])}.$$

Since Z is in $\mathscr{E}(\mathbb{R}, L^p(\Omega, P, \mathbb{X}), \mu)$, we have, for the above ε,

$$\lim_{r\to\infty} \frac{\mu\left\{t\in[-r,r];\, E\big(\|Z(t)\|^p\big) > \varepsilon\eta^p\right\}}{\mu([-r,r])} = 0.$$

We conclude, using [16, Theorem 2.14] (see Remark 1.1.4), and the boundedness of $F(t)$ in $L^p(\Omega, P, \mathbb{Y})$, that $t \to f(t, X(t)) - f(t, Y(t))$ is in $\mathscr{E}(\mathbb{R}, L^p(\Omega, P, \mathbb{Y}), \mu)$.

■

2.2 superposition theorem's for Stepanov almost periodic type functions and processes

In this section, we study some properties of parametric functions, especially Nemytskii's operators $\mathscr{N}(f)(x) := [t \mapsto f(t, x(t))]$ built on $f : \mathbb{R}\times\mathbb{E} \to \mathbb{E}$ in the space of Stepanov (μ-pseudo) almost periodic functions. In the following, we assume that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ with p, q and $r \geq 1$, and we consider the parametric function $f : \mathbb{R}\times\mathbb{E} \to \mathbb{E}$, which satisfies the Lipschitz condition :

(Lip) There exists a nonnegative function $L \in \mathbb{S}^r(\mathbb{R})$ such that

$$d(f(t,u), f(t,v)) \leq L(t)d(u,v),\ \forall t \in \mathbb{R},\ u, v \in \mathbb{E}.$$

Using compactness property of Stepanov almost periodicity given by Danilov (see Theorem 1.2.1 and Corollary 1.2.2), we improve the composition theorem of Stepanov almost periodic functions given in [40, Theorem 2.1]. We show in particular that in order to obtain that $\mathscr{N}(f)$ maps $\mathbb{S}^q\mathrm{AP}(\mathbb{R},\mathbb{E})$ into $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$, the following compactness condition :

(Com) There exists a subset $A \subset \mathbb{R}$ with $\mathrm{meas}(A) = 0$ such that $K := \overline{\{x(t) : t \in \mathbb{R}\backslash A\}}$ is a compact subset of $\mathbb{E}$,

is not necessary. Let us mention that Andres and Pennequin [6, Lemma 3.2] have shown that if $f : \mathbb{E} \to \mathbb{E}$ is a continuous function satisfying, for some $a, b > 0$ and $p, q \geq 1$, the following growth condition :

$$\forall x \in \mathbb{E},\ \|f(x)\|_{\mathbb{E}} \leq a\,\|x\|_{\mathbb{E}}^{p/q} + b,$$

and $g : \mathbb{R} \to \mathbb{E}$ is an $\mathbb{S}^p$AP-function, then the composition $f \circ g$ is an $\mathbb{S}^q$AP-function.
We begin by the following Lemma which identifies the spaces $\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{E},\mathbb{E})$ and $\mathbb{S}^p\mathrm{APU_c}(\mathbb{R}\times\mathbb{E},\mathbb{E})$ under Condition (Lip).

Lemma 2.2.1. *Let $f : \mathbb{R}\times\mathbb{E} \to \mathbb{E}$ be a parametric function satisfying Condition* (Lip). *Then, $f \in \mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{E},\mathbb{E})$ if, and only if, $f \in \mathbb{S}^p\mathrm{APU_c}(\mathbb{R}\times\mathbb{E},\mathbb{E})$.*

The proof of this lemma is very similar to that of Fan et al. [42, Lemma 3.1] which is the analogous result for almost automorphic case.

Now, before giving the superposition theorem in $\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{E},\mathbb{E})$, we need some more notations. Let $u:\mathbb{R}\to\mathbb{E}$ be a mesurable function, let $A\subset\mathbb{R}$ be a measurable set. We denote by $u^{\lfloor A,x_0\rfloor}$ the "truncated" function from $\mathbb{R}$ to $\mathbb{E}$, defined by

$$u^{\lfloor A,x_0\rfloor}(t)=\begin{cases} u(t) & \text{if } \ t\in A\\ x_0 & \text{if } \ t\notin A.\end{cases}$$

We are now ready to present the superposition theorem in $\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{E},\mathbb{E})$:

Theorem 2.2.2. *Let $f\in\mathbb{S}^p\mathrm{AP}(\mathbb{R}\times\mathbb{E},\mathbb{E})$, and assume that f satisfies Condition* (Lip). *Then, for every $u\in\mathbb{S}^q\mathrm{AP}(\mathbb{R},\mathbb{E})$, we have $f(.,u(.))\in\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$.*

Proof : Fix $\varepsilon>0$ and $x_0\in\mathbb{E}$. Let $u\in\mathbb{S}^q\mathrm{AP}(\mathbb{R},\mathbb{E})$. In view of (1.20), we have $u\in M'_p(\mathbb{R},\mathbb{E})\cap \mathbb{S}^0\mathrm{AP}(\mathbb{R},\mathbb{E})$, thus, there exists $\eta>0$ such that $D^d_{\mathbb{S}^q}\big(u^{\lfloor A,x_0\rfloor}(.),x_0\big)\le\varepsilon$ for all measurable set $A\subset\mathbb{R}$ satisfying $\varkappa(A)\le\eta$. For such η, using Corollary 1.2.2, we deduce that there exists a compact subset $\mathscr{K}_{\eta(\varepsilon)}\subset\mathbb{E}$ such that

$$\varkappa\{t\in\mathbb{R},\ x(t)\notin\mathscr{K}_{\eta(\varepsilon)}\}<\eta$$

and

$$D^d_{\mathbb{S}^q}\big(u^{\lfloor T^c_\varepsilon,x_0\rfloor}(.),x_0\big)\le\frac{\varepsilon}{6\,\|L\|_{\mathbb{S}^r}},\tag{2.4}$$

where $T_\varepsilon:=T_{\eta(\varepsilon)}$ is the subset of $\mathbb{R}$ on which $u(t)\in\mathscr{K}_{\eta(\varepsilon)}$ (we exclude for simplicity the trivial case when $\|L\|_{\mathbb{S}^r}=0$). The compactness of $\mathscr{K}_{\eta(\varepsilon)}$ implies that there exist a finite sequence $(x_1,x_2,...,x_n)$ in $\mathscr{K}_{\eta(\varepsilon)}$ such that

$$\mathscr{K}_{\eta(\varepsilon)}\subset\bigcup_{1\le i\le n}B\left(x_i,\frac{\varepsilon}{6\,\|L\|_{\mathbb{S}^r}}\right).\tag{2.5}$$

For $i=1,\dots,n$ and $t\in\mathbb{R}$, let

$$\tau(t)=\begin{cases}0 & \text{if } t\notin T_\varepsilon\\ \min\left\{i\in\{1,\dots,n\};\,d(u(t),x_i)\le\frac{\varepsilon}{6\|L\|_{\mathbb{S}^r}}\right\} & \text{if } t\in T_\varepsilon,\end{cases}$$

and, for $i=0,\dots,n$ and $\xi\in\mathbb{R}$, let

$$A_{i,\xi}=\{t\in[\xi,\xi+1];\ \tau(t)=i\}.$$

By Lemma 2.2.1, we have $f\in\mathbb{S}^p\mathrm{APU_c}(\mathbb{R}\times\mathbb{E},\mathbb{E})$. Since $u\in\mathbb{S}^q\mathrm{AP}(\mathbb{R},\mathbb{E})$, we can choose a common relatively dense set $\mathscr{T}(f,u,\varepsilon)\subset\mathbb{R}$ such that, for $\tau\in\mathscr{T}(f,u,\varepsilon)$,

$$D^d_{\mathbb{S}^q}(u(.+\tau),u(.))\le\frac{\varepsilon}{3\,\|L\|_{\mathbb{S}^r}}\tag{2.6}$$

and

$$\sum_{i=0}^{n}D^d_{\mathbb{S}^p}\Big(f(.+\tau,x_i),f(.,x_i)\Big)\le\frac{\varepsilon}{3}\tag{2.7}$$

for all $\tau \in \mathscr{T}(f,u,\varepsilon)$. Let $\tau \in \mathscr{T}(f,u,\varepsilon)$. We have

$$\begin{aligned}
D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.+\tau))\,,f(.,u(.))\Big) \\
&\le D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.+\tau)),f(.+\tau,u(.))\Big)+D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.)),f(.,u(.))\Big)\\
&\le \|L\|_{\mathbb{S}^r}\,D^d_{\mathbb{S}^q}\Big(u(.+\tau),u(.)\Big)+D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.)),f(.,u(.))\Big)\\
&\le \frac{\varepsilon}{3}+D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.)),f(.,u(.))\Big).
\end{aligned}$$

Now,

$$\begin{aligned}
&D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.)),f(.,u(.))\Big)=\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1} d^p(f(t+\tau,u(t)),f(t,u(t)))\,dt\right)^{1/p}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\sum_{i=0}^n \mathbf{1}_{A_i,\xi}(t)d^p(f(t+\tau,u(t)),f(t+\tau,x_i))\,dt\right)^{1/p}\\
&\quad+\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\sum_{i=0}^n \mathbf{1}_{A_i,\xi}(t)d^p(f(t+\tau,x_i),f(t,x_i))\,dt\right)^{1/p}\\
&\quad+\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}\sum_{i=0}^n \mathbf{1}_{A_i,\xi}(t)d^p(f(t,x_i),f(t,u(t)))\,dt\right)^{1/p}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}L(t+\tau)\sum_{i=0}^n \mathbf{1}_{A_i,\xi}(t)d^p(u(t),x_i)\,dt\right)^{1/p}\\
&\quad+\sum_{i=0}^n D^d_{\mathbb{S}^p}\Big(f(.+\tau,x_i),f(.,x_i)\Big)\\
&\quad+\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}L(t)\sum_{i=0}^n \mathbf{1}_{A_i,\xi}(t)d^p(u(t),x_i)\,dt\right)^{1/p}\\
&\le \sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}L^r(t+\tau)\,dt\right)^{1/r}\left[D^d_{\mathbb{S}^q}\big(u^{\lfloor T^c_\varepsilon,x_0\rfloor}(.),x_0\big)+\left(\int_\xi^{\xi+1}\sum_{i=1}^n \mathbf{1}_{A_i,\xi}(t)d^q(u(t),x_i)\,dt\right)^{1/q}\right]\\
&\quad+\frac{\varepsilon}{3}+\sup_{\xi\in\mathbb{R}}\left(\int_\xi^{\xi+1}L^r(t)\,dt\right)^{1/r}\left[D^d_{\mathbb{S}^q}\big(u^{\lfloor T^c_\varepsilon,x_0\rfloor}(.),x_0\big)+\left(\int_\xi^{\xi+1}\sum_{i=1}^n \mathbf{1}_{A_{i,\xi}}(t)d^q(u(t),x_i)\,dt\right)^{1/q}\right]\\
&\le 2\,\|L\|_{\mathbb{S}^r}\,\frac{\varepsilon}{6\,\|L\|_{\mathbb{S}^r}}+\frac{\varepsilon}{3}=\frac{2\varepsilon}{3}.
\end{aligned}$$

Combining the preceding inequalities, we deduce that

$$D^d_{\mathbb{S}^p}\Big(f(.+\tau,u(.+\tau)),f(.,u(.))\Big)\le\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}=\varepsilon.$$

□

Theorem 2.2.3. *Let μ be a Borel measure on $\mathbb{R}$ satisfying* (1.6) *and Condition* (H)*, and let $p \geq 1$. Assume that $F \in \mathbb{S}^p\mathrm{PAPU_c}(\mathbb{R}\times\mathbb{E},\mathbb{E},\mu)$. Let G be in $\mathbb{S}^p\mathrm{APU_c}(\mathbb{R}\times\mathbb{E},\mathbb{E})$ such that*

$$\Big[(t,x)\mapsto H(t,x)=d\Big(F(t,x),G(t,x)\Big)\Big]\in\mathbb{S}^p\mathscr{E}U_c(\mathbb{R}\times\mathbb{E},\mathbb{R},\mu). \tag{2.8}$$

Assume that Condition (Lip) *holds for F and G. If X is $\mathbb{S}^q$-weighted pseudo almost periodic, then $F(.,X(.))\in\mathbb{S}^p\mathrm{PAP}(\mathbb{R},\mathbb{E},\mu)$.*

Proof : Since $X\in\mathbb{S}^q\mathrm{PAP}(\mathbb{R},\mathbb{E},\mu)$, there exists $Y\in\mathbb{S}^q\mathrm{AP}(\mathbb{R},\mathbb{E})$ such that

$$Z(.)=d\Big(X(.),Y(.)\Big)\in\mathscr{E}_{\mathbb{S}^p}(\mathbb{R},\mathbb{R},\mu). \tag{2.9}$$

To show that $F(.,X(.))\in\mathbb{S}^p\mathrm{PAP}(\mathbb{R},\mathbb{E},\mu)$, it is enough to have

$$d\Big(F(.,X(.)),G(.,Y(.))\Big)\in\mathscr{E}_{\mathbb{S}^p}(\mathbb{R},\mathbb{R},\mu),$$

since by Theorem 2.2.2, the function $G(.,Y(.))\in\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})$. We have

$$d\Big(F(t,X(t)),G(t,Y(t))\Big)\leq d\Big(F(t,Y(t)),G(t,Y(t))\Big)+d\Big(F(t,Y(t)),F(t,X(t))\Big).$$

Clearly, $d\Big(F(.,Y(.)),F(.,X(.)\Big)\in\mathscr{E}_{\mathbb{S}^p}(\mathbb{R},\mathbb{R},\mu)$. Indeed, for every $r>0$,

$$\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^d_{\mathrm{L}^p}\Big(F^b(t,Y^b(t)),F^b(t,X^b(t)\Big)\,d\mu(t)\leq\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|L\|_{\mathbb{S}^r}\,\mathscr{D}^d_{\mathrm{L}^q}\Big(Y^b(t),X^b(t)\Big)\,d\mu(t)$$

and hence, using (2.9), it follows that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^d_{\mathrm{L}^p}\Big(F^b(t,Y^b(t)),F^b(t,X^b(t)\Big)\,d\mu(t)=0.$$

Now, we claim that $d\Big(F(.,Y(.)),G(.,Y(.))\Big)\in\mathscr{E}_{\mathbb{S}^p}(\mathbb{R},\mathbb{R},\mu)$. In fact, let $\varepsilon>0$ and $x_0\in\mathbb{E}$. Since $Y\in\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{E})\subset M'_p(\mathbb{R},\mathbb{E})$, there exists $\delta:=\delta(\varepsilon)>0$ such that $D^d_{\mathbb{S}^q}(Y^{\lfloor A,x_0\rfloor}(.),x_0)\leq\varepsilon$ for all measurable set $A\subset\mathbb{R}$ satisfying $\varkappa(A)\leq\delta$. Thus, using Corollary 1.2.2, we deduce that there exists a compact subset $\mathscr{K}_\delta\subset\mathbb{E}$ such that

$$\varkappa\{t\in\mathbb{R},\ Y(t)\notin\mathscr{K}_\delta\}<\delta$$

and

$$D^d_{\mathbb{S}^q}(Y^{\lfloor T^c_\delta,x_0\rfloor}(.),x_0)\leq\frac{\varepsilon}{4\,\|L\|_{\mathbb{S}^r}} \tag{2.10}$$

where $T_\delta:=T_{\delta(\varepsilon)}$ is the subset of $\mathbb{R}$ on which $Y(t)\in\mathscr{K}_\delta$. The compactness of $\mathscr{K}_{\delta(\varepsilon)}$ implies that there exist $y_1,y_2,\dots,y_m\in\mathscr{K}_\varepsilon$ such that

$$\mathscr{K}_\varepsilon\subset\bigcup_{1\leq i\leq m}B\left(y_i,\frac{\varepsilon}{4\,\|L\|_{\mathbb{S}^r}}\right). \tag{2.11}$$

Hence,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^{d}_{\mathrm{L}^p}\Big(F^b(t,Y^b(t)),G^b(t,Y^b(t)\Big)\,d\mu(t)\\
&\qquad\le \max_{1\le i\le m}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^{d}_{\mathrm{L}^p}\Big(F^b(t,Y^b(t)),F^b(t,y_i)\Big)\,d\mu(t)\\
&\qquad+ \max_{1\le i\le m}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^{d}_{\mathrm{L}^p}\Big(F^b(t,y_i),G^b(t,y_i)\Big)\,d\mu(t)\\
&\qquad+ \max_{1\le i\le m}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^{d}_{\mathrm{L}^p}\Big(G^b(t,y_i),G^b(t,Y^b(t))\Big)\,d\mu(t)\\
&\qquad:=J_1(r)+J_2(r)+J_3(r).
\end{aligned}$$

We have from (2.10) and (2.11) :

$$\begin{aligned}
J_1(r)+J_3(r) &\le 2\,\|L\|_{\mathbb{S}^r}\max_{1\le i\le m}D^{d}_{\mathbb{S}^q}\Big(Y(.),y_i\Big)\\
&\le 2\,\|L\|_{\mathbb{S}^r}\max_{1\le i\le m}\Big\{D^{d}_{\mathbb{S}^q}\Big(Y^{\lfloor T_\delta,y_i\rfloor}(.),y_i\Big)+D^{d}_{\mathbb{S}^q}\Big(Y^{\lfloor T^c_\delta,y_i\rfloor}(.),y_i\Big)\Big\}\le\varepsilon.
\end{aligned}$$

Let us estimate $J_2(r)$. Using the fact that the parametric function H is in $\mathbb{S}^p\mathscr{E}\mathrm{U}_c(\mathbb{R}\times\mathbb{E},\mathbb{R},\mu)$, we have

$$J_2(r)\le\sup_{y\in\mathscr{K}_\varepsilon}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^{d}_{\mathrm{L}^p}\Big(F^b(t,y),G^b(t,y)\Big)\,d\mu(t),$$

from which we deduce that $\lim_{r\to\infty}J_2(r)=0$. Finally, since ε is arbitrary, we obtain that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\mathscr{D}^{d}_{\mathrm{L}^p}\Big(F^b(t,Y^b(t)),G^b(t,Y^b(t)\Big)\,d\mu(t)=0.$$

This proves that $F(.,X(.))\in\mathbb{S}^p\mathrm{PAP}(\mathbb{R},\mathbb{E},\mu)$. □

Deuxième partie

Contribution to stochastic differential equations

CHAPITRE 3

Almost automorphic type solutions for a class of stochastic differential equations with almost automorphic terms

Explicit counterexample to square-mean pseudo almost automorphy At the time of the submission of the paper [9], there are at least 24 papers listed in Mathematical Reviews related to almost automorphy of solutions to stochastic differential equations, which all have been published in 2010 or later. To our knowledge, except for [45, 46, 57], all other papers claim the existence of square-mean pseudo almost automorphic solutions to stochastic differential equations with coefficients having similar properties. We show that a very simple counterexample from [60] contradicts these claims. The other counterexamples given in [60] also contradict these claims.

Example 3.0.4. (Stationary Ornstein-Uhlenbeck process) Let $W = (W(t))_{t\in\mathbb{R}}$ be a standard Brownian motion on the real line. Let $\alpha, \sigma > 0$, and let X be the stationary Ornstein-Uhlenbeck process (see [56]) defined by

$$X(t) = \sqrt{2\alpha\sigma}\int_{-\infty}^{t} e^{-\alpha(t-s)}dW(s). \tag{3.1}$$

Then X is the only L^2-bounded solution of the following SDE, which is a particular case of Equation (3.1) in [12] :

$$dX(t) = -\alpha X(t)\,dt + \sqrt{2\alpha}\sigma\,dW(t).$$

The process X is Gaussian with mean 0, and we have, for all $t \in \mathbb{R}$ and $\tau \geq 0$,

$$\text{Cov}(X(t), X(t+\tau)) = \sigma^2 e^{-\alpha\tau}.$$

Assume that X is square-mean μ-pseudo almost automorphic, for some Borel measure μ on $\mathbb{R}$ satisfying (1.6) and (1.8). Then we can decompose X as

$$X = Y + Z, \quad Y \in \mathrm{AA}(\mathbb{R}, \mathrm{L}^2(\Omega, \mathbb{R})), \quad Z \in \mathscr{E}(\mathbb{R}, \mathrm{L}^2(\Omega, \mathbb{R}), \mu).$$

By Lemma 1.1.2, we can find an increasing sequence (t_n) of real numbers which converges to $+\infty$ such that $(Z(t_n))$ converges to 0 in $\mathrm{L}^2(\Omega, \mathbb{R})$. Then, we can extract a sequence (still denoted by (t_n) for simplicity) such that $(Y(t_n))$ converges in L^2 to a random variable $\widehat{Y}$. Thus $(X(t_n))$ converges in L^2 to $\widehat{Y}$. Necessarily $\widehat{Y}$ is Gaussian with law $\mathscr{N}(0, 2\alpha\sigma^2)$, and $\widehat{Y}$ is $\mathscr{G}$-measurable, where $\mathscr{G} = \sigma(X_{t_n}; n \geq 0)$. Moreover $(X(t_n), \widehat{Y})$ is Gaussian for every n, and we have, for any integer n,

$$\mathrm{Cov}(X(t_n), \widehat{Y}) = \lim_{m \to \infty} \mathrm{Cov}(X(t_n), X(t_{n+m})) = 0,$$

because $(X^2(t))_{t \in \mathbb{R}}$ is uniformly integrable. This proves that $\widehat{Y}$ is independent of $X(t_n)$ for every n, thus $\widehat{Y}$ is independent of $\mathscr{G}$. Thus $\widehat{Y}$ is constant, a contradiction. Thus (3.1) has no square-mean μ-pseudo almost automorphic solution.
Let us show that X is not Weyl-like nor Besicovitch-like square-mean pseudo almost automorphic. It is enough to disprove the Besicovitch sense. Assume that X is Besicovitch-like square-mean pseudo almost automorphic. As before, using Lemma 1.1.2, we can find a sequence (t_n) converging to $+\infty$ and a process $\widehat{Y}$ such that

$$\lim_{n \to \infty} \left\| X(t_n) - \widehat{Y} \right\|_{\mathbb{B}^2} = 0.$$

In particular, $(X(t_n))$ is Cauchy for $\|.\|_{\mathbb{B}^2}$, thus, for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that, for all $n, m \in \mathbb{N}$,

$$(n \geq N(\varepsilon)) \Rightarrow \left\| X(t_n) - X(t_{n+m}) \right\|_{\mathbb{B}^2} \leq \varepsilon. \tag{3.2}$$

But we have

$$\begin{aligned}
&\left\| X(t_n) - X(t_{n+m}) \right\|_{\mathbb{B}^2}^2 \\
&= \limsup_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{E}\left| X(t_n + s) - X(t_{n+m} + s) \right|^2 ds \\
&= \limsup_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \mathrm{E}\big(X^2(t_n + s) + X^2(t_{n+m} + s) - 2X(t_n + s) X(t_{n+m} + s) \big) ds \\
&= \limsup_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \big(\sigma^2 + \sigma^2 - 2\sigma^2 e^{-\alpha(t_{n+m} - t_n)} \big) ds \\
&= 2\sigma^2 \big(1 - e^{-\alpha(t_{n+m} - t_n)} \big).
\end{aligned}$$

For m large, the last term is arbitrarily close to $2\sigma^2$, which contradicts (3.2) for $\varepsilon < 2\sigma^2$. A similar calculation shows that, for any Borel measure μ on $\mathbb{R}$ satisfying (1.6) and (1.8), and for any Borel measure ν on $[0,1]$ satisfying (1.9), the process X is not square-mean $\mathbb{S}^2_\nu$-μ-pseudo almost automorphic.

3.1 Almost automorphic solution of an equation with almost automorphic coefficients

In the sequel, if $\mathbb{X}$ and $\mathbb{Y}$ are metric spaces, we denote $\mathrm{CUB}(\mathbb{X},\mathbb{Y})$ the space of bounded uniformly continuous functions from $\mathbb{X}$ to $\mathbb{Y}$.
We are given two separable Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, and we consider the semilinear stochastic differential equation,

$$dX_t = AX(t)\,dt + f(t,X(t))\,dt + g(t,X(t))\,dW(t),\ t\in\mathbb{R} \tag{3.3}$$

where $A:\mathrm{Dom}(A)\subset\mathbb{H}_2\to\mathbb{H}_2$ is a densely defined closed (possibly unbounded) linear operator, and $f:\mathbb{R}\times\mathbb{H}_2\to\mathbb{H}_2$, and $g:\mathbb{R}\times\mathbb{H}_2\to L(\mathbb{H}_1,\mathbb{H}_2)$ are continuous functions. In this section, we assume that :

(i) $W(t)$ is an $\mathbb{H}_1$-valued Wiener process with nuclear covariance operator Q (we denote by $\mathrm{tr}\,Q$ the trace of Q), defined on a stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in\mathbb{R}},\mathrm{P})$.

(ii) $A:\mathrm{Dom}(A)\to\mathbb{H}_2$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ such that there exists a constant $\delta>0$ with

$$\|S(t)\|_{L(\mathbb{H}_2)}\leq e^{-\delta t}, t\geq 0.$$

(iii) There exists a constant K such that the mappings $f:\mathbb{R}\times\mathbb{H}_2\to\mathbb{H}_2$ and $g:\mathbb{R}\times\mathbb{H}_2\to L(\mathbb{H}_1,\mathbb{H}_2)$ satisfy

$$\|f(t,x)\|_{\mathbb{H}_2}+\|g(t,x)\|_{L(\mathbb{H}_1,\mathbb{H}_2)}\leq K(1+\|x\|_{\mathbb{H}_2}).$$

(iv) The functions f and g are Lipschitz, more precisely there exists a constant K such that

$$\|f(t,x)-f(t,y)\|_{\mathbb{H}_2}+\|g(t,x)-g(t,y)\|_{L(\mathbb{H}_1,\mathbb{H}_2)}\leq K\|x-y\|_{\mathbb{H}_2}$$

for all $t\in\mathbb{R}$ and $x,y\in\mathbb{H}_2$.

(v) $f\in\mathrm{PAAU}_b(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2,\mu)$ and $g\in\mathrm{PAAU}_b(\mathbb{R}\times\mathbb{H}_2,L(\mathbb{H}_1,\mathbb{H}_2),\mu)$ for some given Borel measure μ on $\mathbb{R}$ which satisfies (1.6) and Condition (**H**).

By [16, Theorem 3.5], Condition (**H**) implies that $\mathscr{E}(\mathbb{R},\mathbb{X},\mu)$ and $\mathrm{PAA}(\mathbb{R},\mathbb{X},\mu)$ are translation invariant.
In order to study the weighted pseudo almost automorphy property of solutions of SDEs, we need a result on almost automorphy.

Theorem 3.1.1. *Let the assumptions (4.1) - (4.1) be fulfilled, and assume furthermore the following condition, which is stronger than (4.1) :*
(4.1') $f\in AAU_b(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2)$ *and* $g\in AAU_b(\mathbb{R}\times\mathbb{H}_2,L(\mathbb{H}_1,\mathbb{H}_2))$.
Assume further that $\theta:=\dfrac{K^2}{\delta}\left(\dfrac{1}{2\delta}+\mathrm{tr}\,Q\right)<1$. *Then there exists a unique mild solution* X *to (4.1) in the space* $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ *of bounded uniformly continuous mappings from*

$\mathbb{R}$ to $\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)$. Furthermore, X has a.e. continuous trajectories, and $X(t)$ satisfies, for each $t\in\mathbb{R}$:

$$X(t)=\int_{-\infty}^{t}S(t-s)f\big(s,X(s)\big)ds+\int_{-\infty}^{t}S(t-s)g\big(s,X(s)\big)dW(s). \tag{3.4}$$

If furthermore $\theta' := \frac{4K^2}{\delta}\left(\frac{1}{\delta}+\operatorname{tr}Q\right)<1$, then X is almost automorphic in 2-distribution.

The proof of this theorem is very similar to that of [49, Theorem 3.1], which is the analogous result for SDEs with almost periodic coefficients. Only the almost automorphy part needs to be adapted. Such an adaptation is provided in [57], for SDEs driven by Lévy processes, but only for one-dimensional almost automorphy. We give the proof of this part for the convenience of the reader.
Let us first recall the following result, which is given in a more general form in [28] :

Proposition 3.1.1. *([28, Proposition 3.1-(c)]) Let $\tau\in\mathbb{R}$. Let $(\xi_n)_{0\le n\le\infty}$ be a sequence of square integrable $\mathbb{H}_2$-valued random variables. Let $(f_n)_{0\le n\le\infty}$ and $(g_n)_{0\le n\le\infty}$ be sequences of mappings from $\mathbb{R}\times\mathbb{H}_2$ to $\mathbb{H}_2$ and $L(\mathbb{H}_1,\mathbb{H}_2)$ respectively, satisfying (4.1) and (4.1) (replacing f and g by f_n and g_n respectively, and the constant K being independent of n). For each n, let X_n denote the solution to*

$$X_n(t)=S(t-\tau)\xi_n + \int_{\tau}^{t}S(t-s)f_n\big(s,X_n(s)\big)ds+\int_{\tau}^{t}S(t-s)g_n\big(s,X_n(s)\big)dW(s).$$

Assume that, for every $(t,x)\in\mathbb{R}\times\mathbb{H}_2$,

$$\lim_{n\to\infty}f_n(t,x)=f_\infty(t,x),\ \lim_{n\to\infty}g_n(t,x)=g_\infty(t,x),$$
$$\lim_{n\to\infty}\mathfrak{d}_{\mathrm{BL}}(\mathrm{law}(\xi_n,W),\mathrm{law}(\xi_\infty,W))=0,$$

(the last equality takes place in $\mathscr{M}^{1,+}(\mathbb{H}_2\times\mathrm{C}(\mathbb{R},\mathbb{H}_1))$). Then we have in $\mathrm{C}([\tau,T];\mathbb{H}_2)$, for any $T>\tau$,

$$\lim_{n\to\infty}\mathfrak{d}_{\mathrm{BL}}(\mathrm{law}(X_n),\mathrm{law}(X_\infty))=0.$$

We need also a variant of Gronwall's lemma.

Lemma 3.1.2. *([49, Lemma 3.3]) Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function such that, for every $t\in\mathbb{R}$,*

$$0\le g(t)\le\alpha(t)+\beta_1\int_{-\infty}^{t}e^{-\delta_1(t-s)}g(s)\,ds+\cdots+\beta_n\int_{-\infty}^{t}e^{-\delta_n(t-s)}g(s)\,ds, \tag{3.5}$$

for some locally integrable function $\alpha:\mathbb{R}\to\mathbb{R}$, and for some constants $\beta_1,\dots,\beta_n\ge 0$, and some constants $\delta_1,\dots,\delta_n>\beta$, where $\beta:=\sum_{i=1}^n\beta_i$. We assume that the integrals in the right hand side of (3.5) are convergent. Let $\delta=\min_{1\le i\le n}\delta_i$. Then, for every $\gamma\in]0,\delta-\beta]$ such that $\int_{-\infty}^0 e^{\gamma s}\alpha(s)\,ds$ converges, we have, for every $t\in\mathbb{R}$,

$$g(t)\le\alpha(t)+\beta\int_{-\infty}^{t}e^{-\gamma(t-s)}\alpha(s)\,ds.$$

In particular, if α is constant, we have

$$g(t) \leq \alpha \frac{\delta}{\delta - \beta}.$$

Proof of Theorem 3.1.1 The proof of the existence and uniqueness of a mild solution to (4.1) in $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ is the same as that of Theorem 3.1 in [49] or Theorem 3.3.1 in [**?**].
For the almost automorphy part, let (γ_n') be a sequence in $\mathbb{R}$. Since f and g are almost automorphic, there exists a subsequence (γ_n) and functions $\widehat{f} : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$ and $\widehat{g} : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ such that

$$\lim_{n\to\infty} f(t+\gamma_n, x) = \widehat{f}(t,x), \quad \lim_{n\to\infty} \widehat{f}(t-\gamma_n, x) = f(t,x) \tag{3.6}$$

$$\lim_{n\to\infty} g(t+\gamma_n, x) = \widehat{g}(t,x), \quad \lim_{n\to\infty} \widehat{g}(t-\gamma_n, x) = g(t,x). \tag{3.7}$$

These limits are taken uniformly with respect to x in bounded subsets of $\mathbb{H}_2$.
For each fixed integer n, we consider

$$X_n(t) = \int_{-\infty}^{t} S(t-s) f(s+\gamma_n, X_n(s))\, ds + \int_{-\infty}^{t} S(t-s) g(s+\gamma_n, X_n(s))\, dW(s)$$

the mild solution to

$$dX_n(t) = AX_n(t)dt + f(t+\gamma_n, X_n(t))\, dt + g(t+\gamma_n, X_n(t))\, dW(t)$$

and

$$\widehat{X}(t) = \int_{-\infty}^{t} S(t-s) \widehat{f}(s, \widehat{X}(s))\, ds + \int_{-\infty}^{t} S(t-s) \widehat{g}(s, \widehat{X}(s))\, dW(s)$$

the mild solution to

$$d\widehat{X}(t) = A(t)\widehat{X}(t)dt + \widehat{f}(t, \widehat{X}(t))\, dt + \widehat{g}(t, \widehat{X}(t))\, dW(t).$$

Make the change of variable $\sigma + \gamma_n = s$, the process

$$X(t+\gamma_n) = \int_{-\infty}^{t+\gamma_n} S(t+\gamma_n - s) f(s, X(s))\, ds + \int_{-\infty}^{t+\gamma_n} S(t+\gamma_n - s) g(s, X(s))\, dW(s)$$

satisfies

$$X(t+\gamma_n) = \int_{-\infty}^{t} S(t-s) f(s+\gamma_n, X(s+\gamma_n)) ds + \int_{-\infty}^{t} S(t-s) g(s+\gamma_n, X(s+\gamma_n)) d\tilde{W}_n(s),$$

where $\tilde{W}_n(s) = W(s+\gamma_n) - W(\gamma_n)$ is a Brownian motion with the same distribution as $W(s)$. Thus the process $X(.+\gamma_n)$ has the same distribution as X_n.

Let us show that $X_n(t)$ converges in quadratic mean to $\widehat{X}(t)$ for each fixed $t \in \mathbb{R}$. We have

$$\begin{aligned}
&\mathrm{E}\|X_n(t)-\widehat{X}(t)\|^2\\
&=\mathrm{E}\bigg\|\int_{-\infty}^{t} S(t-s)\Big(f(s+\gamma_n,X_n(s))-\widehat{f}(s,\widehat{X}(s))\Big)\,ds\\
&\quad+\int_{-\infty}^{t} S(t-s)\Big(g(s+\gamma_n,X^n(s))-\widehat{g}(s,\widehat{X}(s))\Big)\,dW(s)\bigg\|^2\\
&\leq 2\,\mathrm{E}\bigg\|\int_{-\infty}^{t} S(t-s)\Big(f(s+\gamma_n,X_n(s))-\widehat{f}(s,\widehat{X}(s))\Big)\,ds\bigg\|^2\\
&\quad+2\,\mathrm{E}\int_{-\infty}^{t} S(t-s)\Big(g(s+\gamma_n,X_n(s))-\widehat{g}(s,\widehat{X}(s))\Big)\,dW(s)\bigg\|^2\\
&\leq 4\,\mathrm{E}\bigg\|\int_{-\infty}^{t} S(t-s)\Big(f(s+\gamma_n,X^n(s))-f(s+\gamma_n,\widehat{X}(s))\Big)\,ds\bigg\|^2\\
&\quad+4\,\mathrm{E}\|\int_{-\infty}^{t} S(t-s)\Big(f(s+\gamma_n,\widehat{X}(s))-\widehat{f}(s,\widehat{X}(s))\Big)\,ds\bigg\|^2\\
&\quad+4\,\mathrm{E}\|\int_{-\infty}^{t} S(t-s)\Big(g(s+\gamma_n,X^n(s))-g(s+\gamma_n,\widehat{X}(s))\Big)\,dW(s)\bigg\|^2\\
&\quad+4\,\mathrm{E}\|\int_{-\infty}^{t} S(t-s)\Big(g(s+\gamma_n,\widehat{X}(s))-\widehat{g}(s,\widehat{X}(s))\Big)\,dW(s)\bigg\|^2\\
&\leq I_1+I_2+I_3+I_4.
\end{aligned}$$

Now, using (4.1), (4.1) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
I_1 &= 4\,\mathrm{E}\bigg\|\int_{-\infty}^{t} S(t-s)\Big(f(s+\gamma_n,X_n(s))-f(s+\gamma_n,\widehat{X}(s))\Big)\,ds\bigg\|^2\\
&\leq 4\,\mathrm{E}\bigg(\int_{-\infty}^{t}\|S(t-s)\|\|f(s+\gamma_n,X_n(s))-f(s+\gamma_n,\widehat{X}(s))\|\,ds\bigg)^2\\
&\leq 4\,\mathrm{E}\bigg(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f(s+\gamma_n,X_n(s))-f(s+\gamma_n,\widehat{X}(s))\|ds\bigg)^2\\
&\leq 4\bigg(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\bigg)\bigg(\int_{-\infty}^{t} e^{-\delta(t-s)}\,\mathrm{E}\|f(s+\gamma_n,X_n(s))-f(s+\gamma_n,\widehat{X}(s))\|^2ds\bigg)\\
&\leq \frac{4K^2}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}\,\mathrm{E}\|X_n(s)-\widehat{X}(s)\|^2ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
I_2 &= 4\,\mathrm{E}\bigg\|\int_{-\infty}^{t} S(t-s)[f(s+\gamma_n,\widehat{X}(s))-\widehat{f}(s,\widehat{X}(s))]ds\bigg\|^2\\
&\leq 4\,\mathrm{E}\bigg(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f(s+\gamma_n,\widehat{X}(s))-\widehat{f}(s,\widehat{X}(s))\|ds\bigg)^2
\end{aligned}$$

$$\leq 4\mathrm{E}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|f(s+\gamma_n,\widehat{X}(s))-\widehat{f}(s,\widehat{X}(s))\|^2 ds\right)$$
$$\leq 4\left(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\right)^2 \sup_s \mathrm{E}\|f(s+\gamma_n,\widehat{X}(s))-\widehat{f}(s,\widehat{X}(s))\|^2$$
$$\leq \frac{4}{\delta^2}\sup_s \mathrm{E}\|f(s+\gamma_n,\widehat{X}(s))-\widehat{f}(s,\widehat{X}(s))\|^2,$$

which converges to 0 as $n\to\infty$ because $\sup_{t\in\mathbb{R}}\mathrm{E}\|\widehat{X}(t)\|^2<\infty$ which implies that $(\widehat{X}(t))_t$ is tight relatively to bounded sets.
Applying Itô's isometry, we get

$$I_3 = 4\mathrm{E}\left\|\int_{-\infty}^{t} S(t-s)\left(g(s+\gamma_n,X_n(s))-g(s+\gamma_n,\widehat{X}(s))\right)dW(s)\right\|^2$$
$$\leq 4\,\mathrm{tr}\,Q\,\mathrm{E}\int_{-\infty}^{t}\|S(t-s)\|^2\,\|g(s+\gamma_n,X_n(s))-g(s+\gamma_n,\widehat{X}(s))\|^2 ds$$
$$\leq 4\,\mathrm{tr}\,Q\int_{-\infty}^{t} e^{-2\delta(t-s)}\,\mathrm{E}\|g(s+\gamma_n,X_n(s))-g(s+\gamma_n,\widehat{X}(s))\|^2 ds$$
$$\leq 4K^2\,\mathrm{tr}\,Q\int_{-\infty}^{t} e^{-2\delta(t-s)}\,\mathrm{E}\|X_n(s)-\widehat{X}(s)\|^2 ds,$$

and

$$I_4 = 4\mathrm{E}\left\|\int_{-\infty}^{t} S(t-s)\left(g(s+\gamma_n,\widehat{X}(s))-\widehat{g}(s,\widehat{X}(s))\right)dW(s)\right\|^2$$
$$\leq 4\,\mathrm{tr}\,Q\,\mathrm{E}\left(\int_{-\infty}^{t}\|S(t-s)\|^2\|g(s+\gamma_n,\widehat{X}(s))-\widehat{g}(s,\widehat{X}(s))\|^2 ds\right)$$
$$\leq 4\,\mathrm{tr}\,Q\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}ds\right)\sup_{s\in\mathbb{R}}\mathrm{E}\|g(s+\gamma_n,\widehat{X}(s))-\widehat{g}(s,\widehat{X}(s))\|^2$$
$$\leq \frac{2\,\mathrm{tr}\,Q}{\delta}\sup_{s\in\mathbb{R}}\mathrm{E}\|g(s+\gamma_n,\widehat{X}(s))-\widehat{g}(s,\widehat{X}(s))\|^2.$$

For the same reason as for I_2, the right hand term goes to 0 as $n\to\infty$.
We thus have

$$\mathrm{E}\|X_n(t)-\widehat{X}(t)\|^2 \leq \alpha_n + \frac{4K^2}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}\,\mathrm{E}\|X_n(s)-\widehat{X}(s)\|^2\,ds$$
$$+4K^2\,\mathrm{tr}\,Q\int_{-\infty}^{t} e^{-2\delta(t-s)}\,\mathrm{E}\|X_n(s)-\widehat{X}(s)\|^2\,ds$$

for a sequence (α_n) such that $\lim_{n\to\infty}\alpha_n=0$. Furthermore, $\beta := \frac{4K^2}{\delta}+4K^2\,\mathrm{tr}\,Q<\delta$. We conclude by Lemma 3.1.2 that

$$\lim_{n\to\infty}\mathrm{E}\|X_n(t)-\widehat{X}(t)\|^2=0,$$

hence $X_n(t)$ converges in distribution to $\widehat{X}(t)$. But, since the distribution of $X_n(t)$ is the same as that of $X(t+\gamma_n)$, we deduce that $X(t+\gamma_n)$ converges in distribution to $\widehat{X}(t)$.

By analogy and using (4.12), (4.13) we can easily prove that $\widehat{X}(t-\gamma_n)$ converges in distribution to $X(t)$.
Note that the sequence $(\|X_n(t)\|^2)$ is uniformly integrable, thus $(\|X(t+\gamma_n)\|^2)$ is uniformly integrable too. As (γ_n') is arbitrary, this implies that the family $(\|X(t)\|^2)_{t\in\mathbb{R}}$ is uniformly integrable, because, if not, there would exist a sequence (γ_n') and $t\in\mathbb{R}$ such that no subsequence of $(\|X(t+\gamma_n')\|^2)$ is uniformly integrable.
We have thus proved that X has almost automorphic one-dimensional 2-distributions.
To prove that X is almost automorphic in 2-distribution, we apply Proposition 3.1.1 : for fixed $\tau\in\mathbb{R}$, let $\xi_n = X(\tau+\gamma_n)$, $f_n(t,x) = f(t+\gamma_n,x)$, $g_n(t,x) = g(t+\gamma_n,x)$. By the foregoing, (ξ_n) converges in distribution to some variable $Y(\tau)$. We deduce that (ξ_n) is tight, and thus (ξ_n,W) is tight also. We can thus choose a subsequence (still noted (γ_n) for simplicity) such that (ξ_n,W) converges in distribution to $(Y(\tau),W)$. Then, by Proposition 3.1.1, for every $T\geq\tau$, $X(.+\gamma_n)$ converges in distribution on $\mathrm{C}([\tau,T];\mathbb{H}_2)$ to the (unique in distribution) solution to

$$Y(t) = S(t-\tau)Y(\tau) + \int_\tau^t S(t-s)f\big(s,Y(s)\big)\,ds + \int_\tau^t S(t-s)g\big(s,Y(s)\big)\,dW(s).$$

Note that Y does not depend on the chosen interval $[\tau,T]$, thus the convergence takes place on $\mathrm{C}(\mathbb{R};\mathbb{H}_2)$. Similarly, $Y_n := Y(.-\gamma_n)$ converges in distribution on $\mathrm{C}(\mathbb{R};\mathbb{H}_2)$ to X. Thus X is almost automorphic in 2-distribution. ∎

3.2 Weighted pseudo almost automorphic solution of an equation with weighted pseudo almost automorphic coefficients

We are now ready to prove our main result.

Theorem 3.2.1. *Let the assumptions (4.1) - (4.1) be fulfilled. Let (f_1,g_1) and (f_2,g_2) be respectively the decompositions of f and g, namely,*

$$\begin{gathered} f = f_1 + f_2, \quad g = g_1 + g_2, \\ f_1 \in AAU_b(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2),\ f_2 \in \mathscr{E}U_b(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2,\mu), \\ g_1 \in AAU_b(\mathbb{R}\times\mathbb{H}_2,L(\mathbb{H}_1,\mathbb{H}_2)),\ g_2 \in \mathscr{E}U_b(\mathbb{R}\times\mathbb{H}_2,L(\mathbb{H}_1,\mathbb{H}_2),\mu). \end{gathered}$$

Assume that f_1 and g_1 satisfy the same growth and Lipschitz conditions (4.1) - (4.1) as f and g respectively, with same coefficient K. Assume furthermore that

$$\theta' := \frac{4K^2}{\delta}\left(\frac{1}{\delta} + \operatorname{tr} Q\right) < 1.$$

Then there exists a unique mild solution X to (4.1) *in the space* $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ *of bounded uniformly continuous mappings from $\mathbb{R}$ to $\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)$, X has a.e. continuous trajectories, and X satisfies* (4.2) *for every $t\in\mathbb{R}$. Furthermore, X is μ-pseudo almost automorphic in*

2-distribution. More precisely, let $Y \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ *be the unique almost automorphic in distribution mild solution to*

$$dY(t) = AY(t)\,dt + f_1(t, Y(t))\,dt + g_1(t, Y(t))\,dW(t),\ t \in \mathbb{R}. \tag{3.8}$$

Then X *has the decomposition*

$$X = Y + Z, \quad Z \in \mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2), \mu\big).$$

The following technical lemma will be used several times.

Lemma 3.2.1. *Let* $\mathfrak{h} \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \mu)$. *Then the function*

$$t \mapsto \left(\int_{-\infty}^{t} e^{-2\delta(t-s)} \mathfrak{h}^2(s)\,ds\right)^{1/2}$$

is also in $\mathscr{E}(\mathbb{R}, \mathbb{R}, \mu)$.

Proof : by Condition **(H)** and [16, Theorem 3.9], we have, for every $u \in \mathbb{R}$,

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} |\mathfrak{h}(t-u)|\,d\mu(t) = 0.$$

We deduce, by Lebesgue's dominated convergence theorem,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_{-\infty}^{t} e^{-2\delta(t-s)} \mathfrak{h}^2(s)\,ds\right)^{1/2} d\mu(t) \\
&\qquad \leq \frac{1}{\big(\mu([-r,r])\big)^{1/2}} \left(\int_{[-r,r]} \int_{-\infty}^{t} e^{-2\delta(t-s)} \mathfrak{h}^2(s)\,ds\,d\mu(t)\right)^{1/2} \\
&\qquad = \frac{1}{\big(\mu([-r,r])\big)^{1/2}} \left(\int_{[-r,r]} \int_{0}^{+\infty} e^{-2\delta u} \mathfrak{h}^2(t-u)\,du\,d\mu(t)\right)^{1/2} \\
&\qquad = \frac{1}{\big(\mu([-r,r])\big)^{1/2}} \left(\int_{0}^{+\infty} e^{-2\delta u} \int_{[-r,r]} \mathfrak{h}^2(t-u)\,d\mu(t)\,du\right)^{1/2} \\
&\qquad \leq \left(\int_{0}^{+\infty} e^{-2\delta u} \|\mathfrak{h}\|_\infty \frac{\int_{[-r,r]} |\mathfrak{h}(t-u)|\,d\mu(t)}{\mu([-r,r])}\,du\right)^{1/2} \\
&\qquad \to 0 \text{ when } r \to +\infty.
\end{aligned}$$

■

Proof of Theorem 4.1.1 The existence and the properties of Y are guaranteed by Theorem 3.1.1.
As in Theorem 3.1.1, the existence and uniqueness of the mild solution X to (4.1) are proved as in [49, Theorem 3.1], using the classical method of the fixed point theorem for the contractive operator L on $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ defined by

$$LX(t) = \int_{-\infty}^{t} S(t-s) f\big(s, X(s)\big) ds + \int_{-\infty}^{t} S(t-s) g\big(s, X(s)\big) dW(s).$$

The solution X defined by (4.2) is thus the limit in $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ of a sequence (X_n) with arbitrary X_0 and, for every n, $X_{n+1}=L(X_n)$. To prove that X is μ-pseudo almost automorphic in 2-distribution we choose a special sequence. Set

$$X_0=Y,\ X_{n+1}=L(X_n),\ Z_n=X_n-Y,\ n\in\mathbb{N}.$$

Let us prove that each Z_n is in $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$. We use some arguments of the proof of [16, Theorem 5.7]. We have, for every $n\in\mathbb{N}$ and every $t\in\mathbb{R}$,

$$\begin{aligned}
Z_{n+1}(t)&=LX_n(t)-Y(t)\\
&=\int_{-\infty}^{t}S(t-s)\big(f(s,X_n(s))-f(s,Y(s))\big)\,ds\\
&\quad+\int_{-\infty}^{t}S(t-s)\big(g(s,X_n(s))-g(s,Y(s))\big)\,dW(s)\\
&\quad+\int_{-\infty}^{t}S(t-s)\big(f(s,Y(s))-f_1(s,Y(s))\big)\,ds\\
&\quad+\int_{-\infty}^{t}S(t-s)\big(g(s,Y(s))-g_1(s,Y(s))\big)\,dW(s)\\
&=\int_{-\infty}^{t}S(t-s)\big(f(s,X_n(s))-f(s,Y(s))\big)\,ds\\
&\quad+\int_{-\infty}^{t}S(t-s)\big(g(s,X_n(s))-g(s,Y(s))\big)\,dW(s)\\
&\quad+\int_{-\infty}^{t}S(t-s)\,f_2(s,Y(s))\,ds+\int_{-\infty}^{t}S(t-s)g_2(s,Y(s))\,dW(s).
\end{aligned}$$

Assume that $Z_n\in\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$. By the Lipschitz condition (4.1),

$$\big(\mathrm{E}\,\|\,f(t,X_n(t))-f(t,Y(t)\|^2\big)^{1/2}\le K\big(\mathrm{E}\,\|Z_n(t)\|^2\big)^{1/2}$$

thus the mapping

$$\mathfrak{f}:t\mapsto\big(\mathrm{E}\,\|\,f(t,X_n(t))-f(t,Y(t)\|^2\big)^{1/2}$$

is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$. The same conclusion holds for

$$\mathfrak{g}:t\mapsto\big(\mathrm{E}\,\|g(t,X_n(t))-g(t,Y(t)\|^2\big)^{1/2}.$$

We get, using Lemma 4.1.5,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\mathrm{E}\left\|\int_{-\infty}^{t}S(t-s)\big(f(s,X_n(s))-f(s,Y(s))\big)\,ds\right\|^2\right)^{1/2}d\mu(t)\\
&\qquad\le\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t}e^{-2\delta(t-s)}\mathfrak{f}^2(s)\,ds\right)^{1/2}d\mu(t)\\
&\qquad\to 0\text{ when }r\to+\infty,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\mathrm{E}\left\|\int_{-\infty}^{t}S(t-s)\big(g(s,X_n(s))-g(s,Y(s))\big)\,dW(s)\right\|^2\right)^{1/2}d\mu(t)&\\
\le (\operatorname{tr}Q)^{1/2}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t}e^{-2\delta(t-s)}\mathfrak{g}^2(s)\,ds\right)^{1/2}d\mu(t)&\\
\to 0 \text{ when } r\to+\infty.&
\end{aligned}$$

To prove that Z_{n+1} is in $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$, there only remains to show that the process $\int_{-\infty}^{t}S(t-s)f_2(s,Y(s))\,ds+\int_{-\infty}^{t}S(t-s)g_2(s,Y(s))\,dW(s)$ belongs to $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$. As Y is almost automorphic in distribution, the family $(\widetilde{Y}(t))=(Y(t+.))_{t\in\mathbb{R}}$ is uniformly tight in $\mathrm{C}_k(\mathbb{R},\mathbb{H}_2)$. In particular, for each $\varepsilon>0$ there exists a compact subset $\mathscr{K}_\varepsilon$ of $\mathrm{C}_k(\mathbb{R},\mathbb{H}_2)$ such that, for every $t\in\mathbb{R}$,

$$\mathrm{P}\left\{\widetilde{Y}(t)\in\mathscr{K}_\varepsilon\right\}\ge 1-\varepsilon.$$

By the Arzelà-Ascoli Theorem (e.g. [41, Theorems 8.2.10 and 8.2.11]), this implies that, for every $\varepsilon>0$, and for every compact interval I of $\mathbb{R}$, there exists a compact subset $K_{\varepsilon,I}$ of $\mathbb{H}_2$ such that, for every $t\in\mathbb{R}$,

$$\mathrm{P}\{(\forall s\in I)\ Y(t+s)\in K_{\varepsilon,I}\}\ge 1-\varepsilon.$$

In particular, for every integer n, we have

$$\mathrm{P}\big\{(\forall t\in[n,n+1])\ Y(t)\in K_{\varepsilon,[0,1]}\big\}\ge 1-\varepsilon.$$

By the uniform continuity property of f_2 and g_2 on $K_{\varepsilon,[0,1]}$, there exists $\eta(\varepsilon)>0$ such that, for all $x,y\in K_{\varepsilon,[0,1]}$,

$$\|x-y\|\le\eta(\varepsilon)\Rightarrow\sup_{t\in\mathbb{R}}\max\big(\|f_2(t,x)-f_2(t,y)\|,\|g_2(t,x)-g_2(t,y)\|\big)\le\varepsilon.$$

We can find a finite sequence $y_1,\dots,y_m$ such that

$$K_{\varepsilon,[0,1]}\subset\bigcup_{i=1}^{m}B(y_i,\eta(\varepsilon)).$$

By [49, Remark 3.6]), the condition $\theta'<1$ ensures that Y is bounded in $\mathrm{L}^p(\mathrm{P},\mathbb{H}_2)$ for some $p>2$ (the same result holds for X, but we do not need it). Note that $f_2=f-f_1$ and $g_2=g-g_1$ satisfy a condition similar to (4.1), which implies that $f_2(.,Y(.))$ and $g_2(.,Y(.))$ are bounded in $\mathrm{L}^p(\mathrm{P},\mathbb{H}_2)$ and $\mathrm{L}^p\big(\mathrm{P},L(\mathbb{H}_1,\mathbb{H}_2)\big)$ respectively. Let

$$\mathfrak{M}_p=\sup_{t\in\mathbb{R}}\max\big(\mathrm{E}\|f_2(.,Y(.))\|^p,\mathrm{E}\|g_2(.,Y(.))\|^p\big)^{2/p}.$$

Let $q=p/(p-2)$. For each integer n, let $\Omega_{\varepsilon,n}$ be the measurable subset of Ω on which $Y(t)\in K_{\varepsilon,[0,1]}$ for all $t\in[n,n+1]$. Let $t\in\mathbb{R}$, and let n be an integer such that $t\in[n,n+1]$. We have

$$
\begin{aligned}
&\left(\mathrm{E}\,\|f_2(t,Y(t))\|^2\right)^{1/2}\\
&\qquad\le \min_{1\le i\le m}\left(\mathrm{E}\big(\mathbf{1}_{\Omega_{\varepsilon,n}}\,\|f_2(t,Y(t))-f(t,y_i)\|^2\big)\right)^{1/2}+\max_{1\le i\le m}\|f_2(t,y_i)\|\\
&\qquad\quad+\left(\mathrm{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,n}}\,\|f_2(t,Y(t))\|^2\big)\right)^{1/2}\\
&\qquad\le \varepsilon+\max_{1\le i\le m}\|f_2(t,y_i)\|+\left(\mathrm{P}\left(\Omega^c_{\varepsilon,n}\right)\right)^{1/q}\left(\mathrm{E}\,\|f_2(t,Y(t))\|^p\right)^{2/p}\\
&\qquad\le \varepsilon+\max_{1\le i\le m}\|f_2(t,y_i)\|+\varepsilon^{1/q}\mathfrak{M}_p.
\end{aligned}
$$

A similar result holds for $\mathrm{E}\,\|g_2(t,Y(t))\|^2$. Let us denote

$$
\begin{aligned}
\mathfrak{E}(\varepsilon)&=\varepsilon+\varepsilon^{1/q}\mathfrak{M}_p,\\
\mathfrak{f}_\varepsilon(t)&=\max_{1\le i\le m}\|f_2(s,y_i)\|,\\
\mathfrak{g}_\varepsilon(t)&=\max_{1\le i\le m}\|g_2(s,y_i)\|.
\end{aligned}
$$

Thanks to the ergodicity of f_2 and g_2, the functions $\mathfrak{f}_\varepsilon$ and $\mathfrak{g}_\varepsilon$ are in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$. We have

$$
\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\mathrm{E}\left\|\int_{-\infty}^t S(t-s)f_2(s,Y(s))\,ds\right.\right.\\
&\qquad\qquad\left.\left.+\int_{-\infty}^t S(t-s)g_2(s,Y(s))\,dW(s)\right\|^2\right)^{1/2}d\mu(t)\\
&\qquad\le\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\left(\int_{-\infty}^t e^{-2\delta(t-s)}\,\mathrm{E}\,\|f_2(s,Y(s))\|^2\,ds\right)^{1/2}\right.\\
&\qquad\quad\left.+(\operatorname{tr}Q)^{1/2}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\,\mathrm{E}\,\|g_2(s,Y(s))\|^2\,ds\right)^{1/2}\right)d\mu(t)\\
&\qquad\le\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\,(\mathfrak{f}_\varepsilon(s)+\mathfrak{E}(\varepsilon))^2\,ds\right)^{1/2}d\mu(t)\\
&\qquad\quad+(\operatorname{tr}Q)^{1/2}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\,(\mathfrak{g}_\varepsilon(s)+\mathfrak{E}(\varepsilon))^2\,ds\right)^{1/2}d\mu(t)\\
&\qquad\le\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\mathfrak{f}^2_\varepsilon(s)\,ds\right)^{1/2}d\mu(t)\\
&\qquad\quad+(\operatorname{tr}Q)^{1/2}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\mathfrak{g}^2_\varepsilon(s)\,ds\right)^{1/2}d\mu(t)\\
&\qquad\quad+\frac{1+(\operatorname{tr}Q)^{1/2}}{\mu([-r,r])}\mathfrak{E}(\varepsilon)\int_{[-r,r]}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\,ds\right)^{1/2}d\mu(t).
\end{aligned}
$$

In the right hand side of the last inequality, the last term is arbitrarily small and both other terms converge to 0 when r goes to $+\infty$, thanks to Lemma 4.1.5. We have thus

proved that Z_{n+1} is in $\mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2), \mu\big)$. We deduce by induction that the sequence (Z_n) lies in $\mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2), \mu\big)$.
Now, the sequence (X_n) converges to X in $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$, thus (Z_n) converges to $Z := X - Y$ in $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$. Let $\varepsilon > 0$, and let n such that

$$\sup_{t\in\mathbb{R}}\big(\mathrm{E}\,\|Z(t) - Z_n(t)\|^2\big)^{1/2} \le \varepsilon.$$

We have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\,\|Z(t)\|^2\big)^{1/2} d\mu(t)\\
&\qquad\le \frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\,\|Z(t) - Z_n(t)\|^2\big)^{1/2} d\mu(t)\\
&\qquad\quad + \frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\,\|Z_n(t)\|^2\big)^{1/2} d\mu(t)\\
&\qquad\le \varepsilon + \frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\,\|Z_n(t)\|^2\big)^{1/2} d\mu(t).
\end{aligned}$$

As ε is arbitrary, this proves that $Z \in \mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2), \mu\big)$. ■

Remark 3.2.2. *We did not use in the proof of Theorem 4.1.1 the hypothesis that* $f_2 \in \mathscr{E}U_b(\mathbb{R}\times\mathbb{H}_2, \mathbb{H}_2, \mu)$ *and* $g_2 \in \mathscr{E}U_b(\mathbb{R}\times\mathbb{H}_2, L(\mathbb{H}_1, \mathbb{H}_2), \mu)$. *Actually, we needed only to assume that, for each* $x \in \mathbb{H}_2$, $f_2(.,x) \in \mathscr{E}(\mathbb{R}, \mathbb{H}_2, \mu)$ *and* $g_2(.,x) \in \mathscr{E}(\mathbb{R}, L(\mathbb{H}_1, \mathbb{H}_2), \mu)$. *By Remark 1.1.3 and the Lipschitz condition, this is equivalent to assume that* $f_2 \in \mathscr{E}U_c(\mathbb{R}\times\mathbb{H}_2, \mathbb{H}_2, \mu)$ *and* $g_2 \in \mathscr{E}U_c(\mathbb{R}\times\mathbb{H}_2, L(\mathbb{H}_1, \mathbb{H}_2), \mu)$.

Theorem 3.2.2. (Weighted pseudo almost periodic solution of an equation with weighted pseudo almost periodic coefficients) *Assume the same hypothesis as in Theorem 4.1.1, and that* f_1 *and* g_1 *are almost periodic with respect to the first variable, uniformly with respect to the second variable in bounded sets. Then (with obvious definitions) the process* Y *of Theorem 4.1.1 is almost periodic in 2-distribution, thus the process* X *is* μ*-pseudo almost periodic in 2-distribution.*

Proof : The proof is exactly the same as that of Theorem 4.1.1, replacing Theorem 3.1.1 by [49, Theorem 3.1]. ■

Chapitre 4

Almost periodic type solution for stochastic differential equations with Stepanov almost periodic type coefficients

4.1 Stepanov almost periodic solutions to stochastic differential equations

Let $(\mathbb{H}_1, \|.\|_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \|.\|_{\mathbb{H}_2})$ be separable Hilbert spaces, and let us denote by $\mathrm{L}(\mathbb{H}_1, \mathbb{H}_2)$ (or $\mathrm{L}(\mathbb{H}_1)$ if $\mathbb{H}_1 = \mathbb{H}_2$) the space of all bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$, and by $\mathrm{L}_2(\mathbb{H}_1, \mathbb{H}_2)$ the space of Hilbert-Schmidt operators from $\mathbb{H}_1$ to $\mathbb{H}_2$.
Recall that $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$, the Banach space of square-mean continuous and L^2-bounded stochastic processes, is endowed with the norm

$$\|X\|_\infty^2 = \sup_t \mathrm{E}\,\|X(t)\|_{\mathbb{H}_2}^2.$$

We consider the semilinear stochastic differential equation

$$dX_t = AX(t)\,dt + F(t, X(t))\,dt + G(t, X(t))\,dW(t),\ t \in \mathbb{R} \tag{4.1}$$

where $A : \mathrm{Dom}(A) \subset \mathbb{H}_2 \to \mathbb{H}_2$ is a densely defined closed (possibly unbounded) linear operator, and $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$, and $G : \mathbb{R} \times \mathbb{H}_2 \to \mathrm{L}_2(\mathbb{H}_1, \mathbb{H}_2)$ are measurable functions (not necessarily continuous). To discuss the existence and uniqueness of almost periodic in $2-$distribution solutions to equation (4.1), we consider the following hypotheses :

(H1) $W(t)$, $t \in \mathbb{R}$, is an $\mathbb{H}_1$-valued Wiener process with nuclear covariance operator Q (we denote by $\mathrm{tr}\,Q$ the trace of Q), defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}}, \mathrm{P})$.

(H2) $A : \mathrm{Dom}(A) \to \mathbb{H}_2$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ such that there exists a constant $\delta > 0$ with

$$\|S(t)\|_{\mathrm{L}(\mathbb{H}_2)} \leq e^{-\delta t}, t \geq 0.$$

(H3) There exists a positive constant M such that the mappings $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$ and $G : \mathbb{R} \times \mathbb{H}_2 \to \mathrm{L}_2(\mathbb{H}_1, \mathbb{H}_2)$ satisfy

$$\|F(t,x)\|_{\mathbb{H}_2} + \|G(t,x)\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)} \leq M(1 + \|x\|_{\mathbb{H}_2})$$

for all $t \in \mathbb{R}$.

(H4) The functions F and G are Lipschitz, more precisely there exists a positive function $K(.) \in \mathbb{S}^p(\mathbb{R})$, $p > 2$, such that

$$\|F(t,x) - F(t,y)\|_{\mathbb{H}_2} + \|G(t,x) - G(t,y)\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)} \leq K(t)\|x - y\|_{\mathbb{H}_2}$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{H}_2$.

(H5) $F \in \mathbb{S}^2\mathrm{AP}(\mathbb{R} \times \mathbb{H}_2, \mathbb{H}_2)$ and $G \in \mathbb{S}^2\mathrm{AP}(\mathbb{R} \times \mathbb{H}_2, \mathrm{L}_2(\mathbb{H}_1, \mathbb{H}_2))$.

4.1.1 Almost periodic solutions in 2-distribution

In order to study the weighted pseudo almost periodicity of solutions to (4.1), we need a result on almost periodicity. In what follows, let $q > 0$ with $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$.

Theorem 4.1.1. *Assume the conditions* (H1) – (H5) *are fulfilled and the constant*

$$\theta_{\mathbb{S}} := \left(\frac{2\,\|K\|^2_{\mathbb{S}^2}}{\delta(1 - e^{-\delta})} + \frac{2\,\|K\|^2_{\mathbb{S}^2}\operatorname{tr} Q}{1 - e^{-2\delta}} \right) < 1.$$

Then there exists a unique mild solution X to (4.1) *in* $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$. *Furthermore, X has a.e. continuous trajectories, and $X(t)$ can be explicitly expressed as follows, for each $t \in \mathbb{R}$:*

$$X(t) = \int_{-\infty}^{t} S(t-s)F\big(s, X(s)\big)ds + \int_{-\infty}^{t} S(t-s)G\big(s, X(s)\big)dW(s). \tag{4.2}$$

If furthermore

$$\theta'_{\mathbb{S}} := \frac{4}{3q\delta}\Big((3\beta_1)^{\frac{q}{2}} + (3\beta_2)^{\frac{q}{2}} \Big) < 1,$$

with

$$\beta_1 := \frac{4}{\delta}\left(\frac{\|K\|^p_{\mathbb{S}^p}}{1 - e^{-\frac{p\delta}{4}}} \right)^{\frac{2}{p}}, \ \beta_2 := 4\operatorname{tr} Q \left(\frac{\|K\|^p_{\mathbb{S}^p}}{1 - e^{-\frac{p\delta}{2}}} \right)^{\frac{2}{p}}$$

then X is almost periodic in 2*-distribution.*

Before giving the proof of Theorem 4.1.1, we need the following lemma and proposition :

Lemma 4.1.2. *Let $K : \mathbb{R} \to \mathbb{R}$ be a nonnegative $\mathbb{S}^p$-bounded (resp. Stepanov almost periodic) function, then the function*

$$\kappa(t) = \int_{-\infty}^{t} e^{-\delta(t-s)} K^p(s)ds$$

is uniformly bounded (resp. Bohr almost periodic), and we have

$$\sup_{t\in\mathbb{R}} \kappa(t) \le \frac{\|K\|_{\mathbb{S}^p}^p}{1-e^{-\delta}}.$$

Proof : The proof is very simple, see for instance [69, 70]. □

The following proposition is based on the application of Komlós's theorem [50].

Proposition 4.1.3. *1. Let* $(\mathbb{U},\Sigma,\lambda)$ *be a* σ*-finite measure space, and let* $\mathbb{B}$ *be a separable Banach space. Let* (F_n') *be a sequence of mappings from* $\mathbb{U}\times\mathbb{B}$ *to* $\mathbb{B}$ *satisfying*

(i) each F_n' *is measurable,*

(ii) for every $u\in\mathbb{U}$*, the sequence* $(F_n'(u,.))$ *is equicontinuous,*

(iii) there exists a measurable mapping $F:\mathbb{U}\times\mathbb{B}\to\mathbb{B}$*, such that*

$$(\forall x\in\mathbb{B})\ \lim_{n\to\infty}\int_{\mathbb{U}}\left\|F_n'(u,x)-F(u,x)\right\|\,d\lambda(u)=0.$$

Then there exists a subsequence (F_n) *of* (F_n')*, a modification* $\tilde{F}$ *of* F*, and a* λ*-negligible subset* $\mathscr{N}$ *of* $\mathbb{U}$ *such that*

$$(\forall x\in\mathbb{B})\ (\forall u\in\mathbb{U}\setminus\mathscr{N})\ \lim_{n\to\infty}F_n(u,x)=\tilde{F}(u,x),$$

and such that, for every $u\in\mathbb{U}\setminus\mathscr{N}$*, the mapping* $F(u,.)$ *is continuous.*
2. With the same hypothesis as in 1., assume now that $\mathbb{U}=\mathbb{R}$ *is the set of real numbers,* Σ *its Borel* σ*-algebra, and* λ *the Lebesgue-measure. Assume furthermore that there exists a sequence* (K_n') *of measurable mappings from* $\mathbb{R}$ *to* $\mathbb{R}^+$*, and a number* $p\ge 1$*, satisfying*

(iv)

$$(\forall n\ge 1)\ (\forall x,y\in\mathbb{B})\ (\forall u\in\mathbb{R})\ \left\|F_n'(u,x)-F_n'(u,y)\right\|\le K_n'(u)\,\|x-y\|,$$

(v)

$$A:=\sup_{n\ge 1}\sup_{u\in\mathbb{R}}\int_u^{u+1}(K_n')^p(v)\,dv<\infty.$$

Then we can extract the subsequence (F_n) *in such a way that there exists a measurable mapping* $K:\mathbb{R}\to\mathbb{R}^+$ *and a* λ*-negligible subset* $\mathscr{N}$ *of* $\mathbb{R}$ *such that*

$$(\forall x,y\in\mathbb{B})\ (\forall u\in\mathbb{U}\setminus\mathscr{N})\ \|F(u,x)-F(u,y)\|\le K(u)\,\|x-y\|, \tag{4.3}$$

with

$$\sup_{u\in\mathbb{R}}\int_u^{u+1}K^p(v)\,dv\le A.$$

Proof : 1. For every $x\in\mathbb{B}$, we can find a subsequence $(F_n^{(x)})$ of (F_n') and a λ-negligible set $\mathscr{N}_x$ such that

$$(\forall u\in\mathbb{U}\setminus\mathscr{N}_x)\ \lim_{n\to\infty}F_n^{(x)}(u,x)=F(u,x).$$

Let $\mathscr{D}$ be a dense countable subset of $\mathbb{B}$. Using a diagonal procedure, we can find a subsequence (F_n) of (F'_n) and a λ-negligible subset $\mathscr{N}$ of $\mathbb{U}$ such that

$$(\forall y \in \mathscr{D})\ (\forall u \in \mathbb{U} \setminus \mathscr{N})\ \lim_{n\to\infty} F_n(u,y) = F(u,y).$$

On the other hand, for every $u \in \mathbb{U}$ and every $x \in \mathbb{B}$, we have, by equicontinuity of $F_n(u,.)$,

$$\lim_{y\to x} \sup_n \|F_n(u,y) - F_n(u,x)\| = 0. \tag{4.4}$$

Let $u \in \mathbb{U} \setminus \mathscr{N}$, and let $x \in \mathbb{B}$. Using the uniformity in (4.4), we deduce, by a classical result on interchange of limits, that, for any $x \in \mathbb{B}$,

$$\lim_{n\to\infty} F_n(u,x) = \lim_{n\to\infty} \lim_{\substack{y\to x\\ y\in\mathscr{D}}} F_n(u,y) = \lim_{\substack{y\to x\\ y\in\mathscr{D}}} \lim_{n\to\infty} F_n(u,y) = \lim_{\substack{y\to x\\ y\in\mathscr{D}}} F(u,y). \tag{4.5}$$

Note that, for $u \in \mathbb{U} \setminus \mathscr{N}$, the calculation (4.5) shows that $f(u,.)$ is continuous on $\mathscr{D}$. Let us define $\tilde{F} : \mathbb{U} \times \mathbb{B} \to \mathbb{B}$ by

$$\tilde{F}(u,x) = \begin{cases} \lim_{n\to\infty} F_n(u,x) = \lim_{\substack{y\to x\\ y\in\mathscr{D}}} F(u,y) & \text{for } u \in \mathbb{U}\setminus\mathscr{N} \text{ and } x \in \mathbb{B},\\ 0 & \text{for } u \in \mathscr{N} \text{ and } x \in \mathbb{B}. \end{cases}$$

This definition is consistent, thanks to (4.5). Furthermore, $\tilde{F}(u,.)$ is continuous on $\mathbb{B}$ for every $u \in \mathbb{U}$. Finally, since $\tilde{F}(u,y) = F(u,y)$ for all $(u,y) \in (\mathbb{U}\setminus\mathscr{N}) \times \mathscr{D}$, we have, for any $x \in \mathbb{B}$,

$$\begin{aligned}\int_{\mathbb{U}} &\left\|\tilde{F}(u,x) - F(u,x)\right\| d\lambda(u)\\ &\le \lim_{\substack{y\to x\\ y\in\mathscr{D}}} \left(\int_{\mathbb{U}} \left\|F_n(u,y) - \tilde{F}(u,x)\right\| d\lambda(u) + \int_{\mathbb{U}} \|F_n(u,y) - F(u,x)\|\, d\lambda(u) \right) = 0,\end{aligned}$$

which proves that $\tilde{F}(u,x) = F(u,x)$ for λ-almost every $u \in \mathbb{U}$.

2. By an application of Komlós's theorem [50] on each interval $[k,k+1]$, where k is an integer, and using a diagonal procedure, we can extract a subsequence (K_n) of (K'_n) and a mapping $K : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} K_j^p(u) = K^p(u) \quad \text{for } \lambda\text{-a.e. } u \in \mathbb{R},$$

and such that this almost sure Cesàro convergence holds true for any further subsequence of (K_n) (the negligible set on which the convergence does not hold depends on the subsequence). We can thus ask for the sequences (F_n) and (K_n) to have the same indices. Denote, for $n \geq 1$,

$$g_n = \frac{1}{n}\sum_{j=1}^{n} F_j, \quad L_n = \left(\frac{1}{n}\sum_{j=1}^{n} K_j^p\right)^{1/p}.$$

There exists a λ-negligible subset $\mathscr{N}$ of $\mathbb{R}$ such that

$$(\forall x \in \mathbb{B})\ (\forall u \in \mathbb{R} \setminus \mathscr{N})\ \lim_{n\to\infty} g_n(u,x) = F(u,x) \text{ and } \lim_{n\to\infty} L_n(u) = K(u). \tag{4.6}$$

On the other hand, by the triangular inequality, we have also :

$$(\forall n \geq 1)\ (\forall x,y \in \mathbb{B})\ (\forall u \in \mathbb{R})\ \|g_n(u,x) - g_n(u,y)\| \leq L_n(u)\,\|x-y\|. \tag{4.7}$$

We deduce (4.3) from (4.6) and (4.7). Furthermore, by Fatou's lemma, we have

$$\sup_{u\in\mathbb{R}} \int_u^{u+1} K^p(v)\,dv \leq \sup_{u\in\mathbb{R}} \liminf_{n\to\infty} \int_u^{u+1} L_n^p(v)\,dv \leq A.$$

□

We are now ready to prove Theorem 4.1.1.
Proof of Theorem 4.1.1 Clearly, the process

$$X(t) = \int_{-\infty}^t T(t-s)F\big(s,X(s)\big)ds + \int_{-\infty}^t T(t-s)G\big(s,X(s)\big)dW(s)$$

satisfies

$$X(t) = T(t-a)X(a) + \int_a^t T(t-s)F\big(s,X(s)\big)ds + \int_a^t T(t-s)G\big(s,X(s)\big)dW(s)$$

for all $t \geq a$ for each $a \in \mathbb{R}$, and hence X is a mild solution to (4.1).
We introduce an operator Γ by

$$\Gamma X(t) = \int_{-\infty}^t T(t-s)F\big(s,X(s)\big)ds + \int_{-\infty}^t T(t-s)G\big(s,X(s)\big)dW(s).$$

First step. Let us show that Γ has a unique fixed point. For this purpose we need to show that Γ maps $\mathbb{S}^2\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ into $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$. Let $X \in \mathbb{S}^2\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$. Put $\Gamma = \Gamma_1 + \Gamma_2$, where

$$(\Gamma_1 X)(t) = \int_{-\infty}^t T(t-s)F\big(s,X(s)\big)ds$$

and

$$(\Gamma_2 X)(t) = \int_{-\infty}^t T(t-s)G\big(s,X(s)\big)dW(s).$$

Using conditions (H2) and (H4), the functions F and G satisfy the properties $f(.) := F(.,X(.)) \in \mathbb{S}^2\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ and $g(.) := G(.,X(.)) \in \mathbb{S}^2\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)\big)$. Let us introduce the following processes, for each $n \geq 1$,

$$(\Gamma_{1,n} X)(t) = \int_{t-n}^{t-n+1} T(t-s)f(s)ds$$

and

$$(\Gamma_{2,n} X)(t) = \int_{t-n}^{t-n+1} T(t-s)g(s)dW(s).$$

Clearly, for each n, $\Gamma_{1,n}X \in \mathrm{C}(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2))$. Likewise for $\Gamma_{2,n}X$, for which the continuity is a property of the stochastic integral. To show the boundedness of $\Gamma_{1,n}X$ and $\Gamma_{2,n}X$ for each fixed $n \geq 1$, we use standard arguments. By Hölder's inequality, we have, for any $t \in \mathbb{R}$, and $n \geq 1$

$$\begin{aligned}
\mathrm{E}\left\|(\Gamma_{1,n}X)(t)\right\|_{\mathbb{H}_2}^2 &\leq \mathrm{E}\left(\int_{t-n}^{t-n+1} \|T(t-s)\| \, \|f(s)\|_{\mathbb{H}_2} \, ds\right)^2 \\
&\leq \delta^{-1} \int_{t-n}^{t-n+1} e^{-\delta(t-s)} \mathrm{E}\,\|f(s)\|_{\mathbb{H}_2}^2 \, ds \\
&\leq \delta^{-1} e^{-\delta(n-1)} \int_{t-n}^{t-n+1} \mathrm{E}\,\|f(s)\|_{\mathbb{H}_2}^2 \, ds,
\end{aligned}$$

which leads to

$$\left\|\Gamma_{1,n}X\right\|_\infty^2 \leq \delta^{-1} e^{-\delta(n-1)} \, \|f\|_{\mathbb{S}^2}^2 .$$

Since the series $\sum_{n=1}^{\infty} e^{-2\delta(n-1)} \, \|f\|_{\mathbb{S}^2}^2$ is convergent, it follows that

$$\Gamma_1 X := \sum_{n=1}^{\infty} \Gamma_{1,n}X \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big). \tag{4.8}$$

By Itô's isometry, we have for every $t \in \mathbb{R}$ and $n \geq 1$,

$$\begin{aligned}
\mathrm{E}\left\|(\Gamma_{2,n}X)(t)\right\|_{\mathbb{H}_2}^2 &= \operatorname{tr} Q \int_{t-n}^{t-n+1} \mathrm{E}\,\|T(t-s)\|^2 \, \|g(s)\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)}^2 \, ds \\
&\leq \operatorname{tr} Q \int_{t-n}^{t-n+1} e^{-2\delta(t-s)} \mathrm{E}\,\|g(s)\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)}^2 \, ds \\
&\leq \operatorname{tr} Q e^{-2\delta(n-1)} \, \|g\|_{\mathbb{S}^2}^2 .
\end{aligned}$$

This shows that $\Gamma_{2,n}X \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ for each $n \geq 1$. Since $\sum_{n=1}^{\infty} e^{-2\delta(n-1)} < +\infty$, the series $\sum_{n=1}^{\infty} (\Gamma_{2,n}X)(t)$ is uniformly convergent on $\mathbb{R}$. Thus

$$\Gamma_2 X := \sum_{n=1}^{\infty} \Gamma_{2,n}X \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big). \tag{4.9}$$

From (4.8) and (4.9), we deduce that Γ maps $\mathbb{S}^2\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ into $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$.
Let us show that Γ is a contraction operator. We have, for any $t \in \mathbb{R}$ and $X, Y \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$,

$$\begin{aligned}
\mathrm{E}\,\|(\Gamma X)(t) - (\Gamma Y)(t)\|_{\mathbb{H}_2}^2 \leq& 2\,\mathrm{E}\left(\int_{-\infty}^{t} e^{-\delta(t-s)} \|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2} ds\right)^2 \\
&+ 2\,\mathrm{E}\left\|\int_{-\infty}^{t} T(t-s)[G(s, X(s)) - G(s, Y(s))] dW(s)\right\|_{\mathbb{H}_2}^2 \\
=& I_1(t) + I_2(t).
\end{aligned}$$

Let us estimate $I_1(t)$. Using (H4), Cauchy-Schwartz inequality, and Lemma 4.1.2, we obtain :

$$I_1(t) \leq 2\left(\int_{-\infty}^{t} e^{-\delta(t-s)} ds\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)} \mathrm{E}\,\|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2}^2 ds\right)$$

$$\leq \frac{2}{\delta}\left(\int_{-\infty}^{t} e^{-\delta(t-s)}K^2(s)\,\mathrm{E}\,\|X(s)-Y(s)\|_{\mathbb{H}_2}^2 ds\right)$$
$$\leq \frac{2}{\delta}\left(\sup_{s\in\mathbb{R}}\mathrm{E}\,\|X(s)-Y(s)\|_{\mathbb{H}_2}^2\right)\left(\int_{-\infty}^{t} e^{-\delta(t-s)}K^2(s)ds\right)$$
$$\leq \frac{2\,\|K\|_{\mathbb{S}^2}^2}{\delta(1-\exp(-\delta))}\sup_{s\in\mathbb{R}}\mathrm{E}\,\|X(s))-Y(s))\|_{\mathbb{H}_2}^2.$$

For $I_2(t)$, using again (H4), Lemma 4.1.2, and Itô's isometry we get :

$$\begin{aligned}I_2(t) &\leq 2\,\mathrm{tr}\,Q\int_{-\infty}^{t} e^{-2\delta(t-s)}\,\mathrm{E}\,\|G(s,X(s))-G(s,Y(s))\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)}^2 ds\\
&\leq 2\,\mathrm{tr}\,Q\int_{-\infty}^{t} e^{-2\delta(t-s)}K^2(s)\,\mathrm{E}\,\|X(s)-Y(s)\|_{\mathbb{H}_2}^2 ds\\
&\leq 2\,\mathrm{tr}\,Q\left(\sup_{s\in\mathbb{R}}\mathrm{E}\,\|X(s)-Y(s)\|_{\mathbb{H}_2}^2\right)\left(\int_{-\infty}^{t} e^{-2\delta(t-s)}K^2(s)ds\right)\\
&\leq \frac{2\,\|K\|_{\mathbb{S}^2}^2\,\mathrm{tr}\,Q}{1-\exp(-2\delta)}\left(\sup_{s\in\mathbb{R}}\mathrm{E}\,\|X(s)-Y(s)\|_{\mathbb{H}_2}^2\right).\end{aligned}$$

We thus have

$$\|\Gamma X-\Gamma Y\|_\infty^2 \leq \left(\frac{2\,\|K\|_{\mathbb{S}^2}^2}{\delta(1-\exp(-\delta))}+\frac{2\,\|K\|_{\mathbb{S}^2}^2\,\mathrm{tr}\,Q}{1-\exp(-2\delta)}\right)\|X-Y\|_\infty^2 = \theta_{\mathbb{S}}\|X-Y\|_\infty^2.$$

Consequently, as $\theta_{\mathbb{S}} < 1$, we deduce that Γ is a contraction operator, hence there exists a unique mild solution to (4.1) in $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$. By [29, Theorem 7.2]), almost all trajectories of this solution are continuous.

Second step. Let us show that X is almost periodic in distribution. For this purpose, we prove first that X is almost periodic in one distribution. We use Bochner's double sequences criterion. Since $F \in \mathbb{S}^2\mathrm{AP}(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2)$ and $G \in \mathbb{S}^2\mathrm{AP}(\mathbb{R}\times\mathbb{H}_2,\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2))$, we deduce, by Proposition 1.2.4, that there exist subsequences $(\alpha_n^{'}) \subset (\alpha_n^{''})$ and $(\beta_n^{'}) \subset (\beta_n^{''})$ with same indexes (and independent of x), and functions $F^\infty \in \mathbb{S}^2\mathrm{AP}(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2)$ and $G^\infty \in \mathbb{S}^2\mathrm{AP}(\mathbb{R}\times\mathbb{H}_2,\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2))$ such that for every $t\in\mathbb{R}$ and $x\in\mathbb{H}_2$

$$\begin{aligned}\lim_{n\to\infty}\int_t^{t+1}&\left\|F(s+\alpha_n^{'}+\beta_n^{'},x)-F^\infty(s,x)\right\|_{\mathbb{H}_2}^2 ds\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\int_t^{t+1}\left\|F(s+\alpha_n^{'}+\beta_m^{'},x)-F^\infty(s,x)\right\|_{\mathbb{H}_2}^2 ds = 0, \quad (4.10)\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\int_t^{t+1}&\left\|G(s+\alpha_n^{'}+\beta_n^{'},x)-G^\infty(s,x)\right\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)}^2 ds\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\int_t^{t+1}\left\|G(s+\alpha_n^{'}+\beta_n^{'},x)-G^\infty(s,x)\right\|_{\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2)}^2 ds = 0. \quad (4.11)\end{aligned}$$

These limits exist also uniformly with respect to $t\in\mathbb{R}$.
Thanks to Proposition 4.1.3, we obtain the following interesting properties :

– The functions F^{∞} and G^{∞} satisfy similar conditions as (H3) and (H4).
– there are subsequences of (α_n) (resp. (β_n)), still noted (for simplicity) by (α_n) (resp. (β_n)) and Lebesgue-negligible subset $\mathscr{N}$ of $\mathbb{R}$ such that for all $s \in \mathbb{R} \setminus \mathscr{N}$ and every $x \in \mathbb{H}_2$

$$\lim_{n\to\infty} F(s+\alpha_n+\beta_n,x) = \lim_{n\to\infty}\lim_{m\to\infty} F(s+\alpha_n+\beta_m,x) = F^{\infty}(s,x), \tag{4.12}$$

$$\lim_{n\to\infty} G(s+\alpha_n+\beta_n,x) = \lim_{n\to\infty}\lim_{m\to\infty} G(s+\alpha_n+\beta_m,x) = G^{\infty}(s,x). \tag{4.13}$$

We now set $\gamma_n = \alpha_n + \beta_n$ and consider the sequence of operators, defined, for each $n \geq 1$, by

$$(\Gamma^n X)(t) = \int_{-\infty}^{t} T(t-s)F(s+\gamma_n,X(s))ds + \int_{-\infty}^{t} T(t-s)G(s+\gamma_n,X(s))dW(s).$$

Let Γ^{∞} be the operator defined by

$$(\Gamma^{\infty} X)(t) = \int_{-\infty}^{t} T(t-s)F^{\infty}(s,X(s))ds + \int_{-\infty}^{t} T(t-s)G^{\infty}(s,X(s))dW(s).$$

Using the same reasoning as in the first step, we deduce that, for each $n \geq 1$, Γ^n maps $\mathbb{S}^2\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ into $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$ and it is a contraction operator, with contraction constant equal to $\theta_{\mathbb{S}}$. It follows, by an application of the fixed point theorem, that there exists a process

$$X^n(t) = \int_{-\infty}^{t} T(t-s)F(s+\gamma_n,X^n(s))ds + \int_{-\infty}^{t} T(t-s)G(s+\gamma_n,X^n(s))dW(s)$$

which is the fixed point of Γ^n and also the mild solution to

$$dX(t) = AX(t)dt + F(t+\gamma_n,X(t))dt + G(t+\gamma_n,X(t))dW(t).$$

Moreover, thanks to Proposition 4.1.3, the mappings F^{∞} and G^{∞} satisfy similar conditions as (H3) and (H4). Hence, the fixed point theorem applied on Γ^{∞} ensures the existence of a process X^{∞}, satisfying the integral equation

$$X^{\infty}(t) = \int_{-\infty}^{t} T(t-s)F^{\infty}(s,X^{\infty}(s))ds + \int_{-\infty}^{t} T(t-s)G^{\infty}(s,X^{\infty}(s))dW(s),$$

that is, X^{∞} is a mild solution to

$$dX(t) = AX(t)dt + F^{\infty}(t,X(t))dt + G^{\infty}(t,X(t))dW(t).$$

Make the change of variable $\sigma + \gamma_n = s$, the process

$$X(t+\gamma_n) = \int_{-\infty}^{t+\gamma_n} T(t+\gamma_n-\sigma)F(\sigma,X(\sigma))d\sigma + \int_{-\infty}^{t+\gamma_n} T(t+\gamma_n-\sigma)G(\sigma,X(\sigma))dW(\sigma)$$

becomes

$$X(t+\gamma_n) = \int_{-\infty}^{t} T(t-s)F(s+\gamma_n,X(s+\gamma_n))ds + \int_{-\infty}^{t} T(t-s)G(s+\gamma_n,X(s+\gamma_n))d\tilde{W}_n(s),$$

where $\tilde{W}_n(s) = W(s+\gamma_n) - W(\gamma_n)$ is a Brownian motion with the same distribution as $W(s)$. We deduce that the process $X(t+\gamma_n)$ has the same distribution as $X^n(t)$.
Let us show that $X^n(t)$ converges in quadratic mean to $X^\infty(t)$ for each fixed $t \in \mathbb{R}$. We have, by the triangular inequality,

$$\begin{aligned}
\mathrm{E}\|X^n(t) - X^\infty(t)\|^2 =& \mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\gamma_n, X^n(s)) - F^\infty(s, X^\infty(s))]ds \\
&+ \int_{-\infty}^t T(t-s)[G(s+\gamma_n, X^n(s)) - G^\infty(s, X^\infty(s))]dW(s)\|^2 \\
\leq& 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^\infty(s))]ds\|^2 \\
&+ 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^\infty(s))]dW(s)\|^2 \\
&+ 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))]ds\|^2 \\
&+ 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[G(s+\gamma_n, X^\infty(s)) - G^\infty(s, X^\infty(s))]dW(s)\|^2 \\
\leq& I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t).
\end{aligned}$$

Now, using (H2), (H4), Hölder's inequality, and Lemma 4.1.2, we obtain

$$\begin{aligned}
I_1^n(t) &= 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^\infty(s))]ds\|^2 \\
&\leq 4\mathrm{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\|F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^\infty(s))\|ds\right)^2 \\
&\leq \frac{4}{\delta}\int_{-\infty}^t e^{-\delta(t-s)}K^2(s+\gamma_n)\,\mathrm{E}\|X^n(s) - X^\infty(s)\|^2 ds \\
&\leq \frac{4}{\delta}\Big(\int_{-\infty}^t e^{-\frac{p\delta}{4}(t-s)}K^p(s+\gamma_n)ds\Big)^{\frac{2}{p}}\Big(\int_{-\infty}^t e^{-\frac{q\delta}{4}(t-s)}\big(\mathrm{E}\|X^n(s) - X^\infty(s)\|^2\big)^{\frac{q}{2}}ds\Big)^{\frac{2}{q}} \\
&\leq \frac{4}{\delta}\left(\frac{\|K\|_{\mathbb{S}^p}^p}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{2}{p}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{2}(t-s)}\big(\mathrm{E}\|X^n(s) - X^\infty(s)\|^2\big)^{\frac{q}{2}}ds\right)^{\frac{2}{q}}.
\end{aligned}$$

For $I_2^n(t)$, using Itô's isometry, Hölder's inequality, and Lemma 4.1.2, we get

$$\begin{aligned}
I_2^n(t) &= 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^\infty(s))]dW(s)\|^2 \\
&\leq 4\,\mathrm{tr}\,Q\,\mathrm{E}\int_{-\infty}^t \|T(t-s)\|^2\|G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^\infty(s))\|^2 ds \\
&\leq 4\,\mathrm{tr}\,Q\int_{-\infty}^t e^{-2\delta(t-s)}K^2(s+\gamma_n)\,\mathrm{E}\|X^n(s) - X^\infty(s)\|^2 ds \\
&\leq 4\,\mathrm{tr}\,Q\left(\int_{-\infty}^t e^{-\frac{p\delta}{2}(t-s)}K^p(s+\gamma_n)ds\right)^{\frac{2}{p}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{2}(t-s)}\big(\mathrm{E}\|X^n(s) - X^\infty(s)\|^2\big)^{\frac{q}{2}}ds\right)^{\frac{2}{q}} \\
&\leq 4\,\mathrm{tr}\,Q\left(\frac{\|K\|_{\mathbb{S}^p}^p}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{2}{p}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{2}(t-s)}\big(\mathrm{E}\|X^n(s) - X^\infty(s)\|^2\big)^{\frac{q}{2}}ds\right)^{\frac{2}{q}}.
\end{aligned}$$

Let us show that $I_3^n(t)$ and $I_4^n(t)$ go to 0 as n goes to infinity.
For any $r \in \mathbb{R}$, since $X^\infty \in \mathrm{CUB}\Big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\Big)$, the family

$$\Big(\|X^\infty(s)\|^2\Big)_{r \le s \le r+1}$$

is uniformly integrable, by the converse to Vitali's theorem. By the growth condition satisfied by F and F^∞, this shows that the family

$$(U_{s,n}) := \Big(\|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\|^2\Big)_{r \le s \le r+1, n \ge 1}$$

is uniformly integrable. By La Vallée Poussin's criterion, there exists a non-negative increasing convex function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t \to \infty} \frac{\Phi(t)}{t} = +\infty$ and $\sup_{s,n} \mathrm{E}(\Phi(U_{s,n})) < +\infty$. We thus have

$$\sup_n \mathrm{E} \int_r^{r+1} \Phi\big(U_{s,n}\big)\, ds < +\infty,$$

which prove that the family $(U_{\cdot,n})_{n \ge 1}$ is uniformly integrable with respect to the probability measure $\mathrm{P} \otimes \lambda$ on $\Omega \times [r, r+1]$, where λ denotes Lebesgue's measure. This proves that, for any $r \in \mathbb{R}$,

$$\lim_{n \to +\infty} \left(\mathrm{E} \left(\int_r^{r+1} \|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\|^2 ds \right) \right)^{1/2} = 0. \tag{4.14}$$

Let $t \ge 0$. Since $X^\infty \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$, and thanks to the growth condition satisfied by F, the sequence

$$\left(\mathrm{E} \int_{t-k}^{t-k+1} \|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\|^2 ds\right)_{k \ge 1, n \ge 0}$$

is bounded. We can thus find an integer $N(t, \eta)$ such that, for any $n \ge 0$,

$$\left(\sum_{k > N(t,\eta)} e^{-\delta(k-1)} \mathrm{E} \int_{t-k}^{t-k+1} \|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\|^2 ds \right)^{1/2} \le \eta. \tag{4.15}$$

Using (4.15), we get

$$\begin{aligned}
\sqrt{I_3^n(t)} \le& 2 \left(\mathrm{E} \left(\int_{-\infty}^t \|T(t-s)\| \|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\| ds \right)^2 \right)^{1/2} \\
\le& 2 \sum_{k=1}^{N(t,\eta)} \left(\mathrm{E} \left(\int_{t-k}^{t-k+1} e^{-\delta(t-s)} \|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\| ds \right)^2 \right)^{1/2} \\
&+ 2 \sum_{N(t,\eta)+1}^{\infty} \left(\mathrm{E} \left(\int_{t-k}^{t-k+1} e^{-\delta(t-s)} \|F(s+\gamma_n, X^\infty(s)) - F^\infty(s, X^\infty(s))\| ds \right)^2 \right)^{1/2}
\end{aligned}$$

$$\leq 2\sum_{k=1}^{N(t,\eta)} e^{-\delta(k-1)}\left[\mathrm{E}\left(\int_{t-k}^{t-k+1}\|F(s+\gamma_n,X^\infty(s))-F^\infty(s,X^\infty(s))\|ds\right)^2\right]^{1/2}+2\eta. \tag{4.16}$$

Since the sum in (4.16) is finite and η is arbitrary, we deduce from (4.14) that

$$\lim_{n\to+\infty} I_3^n(t)=0.$$

For $I_4^n(t)$, applying Itô's isometry, we obtain

$$\begin{aligned} I_4^n(t) &= 4\mathrm{E}\|\int_{-\infty}^t T(t-s)[G(s+\gamma_n,X^\infty(s))-G^\infty(s,X^\infty(s))]dW(s)\|^2\\ &\leq 4\,\mathrm{tr}\,Q\,\mathrm{E}\left(\int_{-\infty}^t e^{-2\delta(t-s)}\|G(s+\gamma_n,X^\infty(s))-G^\infty(s,X^\infty(s))\|^2ds\right). \end{aligned}$$

For the same reason as for $I_3^n(t)$ and by (4.13), $I_4^n(t)$ goes to 0 as $n\to\infty$. Now, let us define the following quantities :

$$\alpha_n(t):=I_3^n(t)+I_4^n(t),\ \beta_1:=\frac{4}{\delta}\left(\frac{\|K\|_{\mathbb{S}^p}^p}{1-e^{-\frac{p\delta}{4}}}\right)^{\frac{2}{p}},\ \beta_2:=4\,\mathrm{tr}\,Q\left(\frac{\|K\|_{\mathbb{S}^p}^p}{1-e^{-\frac{p\delta}{2}}}\right)^{\frac{2}{p}}.$$

From the above, we have

$$\begin{aligned} \mathrm{E}\|X^n(t)-X^\infty(t)\|^2 \leq \alpha_n(t)&+\beta_1\left(\int_{-\infty}^t e^{-\frac{q\delta}{4}(t-s)}\left(\mathrm{E}\|X^n(s)-X^\infty(s)\|^2\right)^{\frac{q}{2}}ds\right)^{\frac{2}{q}}\\ &+\beta_2\left(\int_{-\infty}^t e^{-\frac{q\delta}{2}(t-s)}\left(\mathrm{E}\|X^n(s)-X^\infty(s)\|^2\right)^{\frac{q}{2}}ds\right)^{\frac{2}{q}}. \end{aligned}$$

By convexity of the mapping $u\mapsto u^{\frac{q}{2}}$ defined on $\mathbb{R}^+$, we get

$$\begin{aligned} \left(\mathrm{E}\|X^n(t)-X^\infty(t)\|^2\right)^{\frac{q}{2}} \leq \frac{1}{3}\left(3\alpha_n(t)\right)^{\frac{q}{2}}&+\frac{1}{3}\left(3\beta_1\right)^{\frac{q}{2}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{4}(t-s)}\left(\mathrm{E}\|X^n(s)-X^\infty(s)\|^2\right)^{\frac{q}{2}}ds\right)\\ &+\frac{1}{3}\left(3\beta_2\right)^{\frac{q}{2}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{2}(t-s)}\left(\mathrm{E}\|X^n(s)-X^\infty(s)\|^2\right)^{\frac{q}{2}}ds\right). \end{aligned}$$

Since $\theta'_{\mathbb{S}}=\frac{4}{3q\delta}\left(\left(3\beta_1\right)^{\frac{q}{2}}+\left(3\beta_2\right)^{\frac{q}{2}}\right)<1$, we obtain, by Gronwall's Lemma as in [49, Lemma 3.3],

$$\left(\mathrm{E}\|X^n(t)-X^\infty(t)\|^2\right)^{\frac{q}{2}} \leq \frac{1}{3}\left(3\alpha_n(t)\right)^{\frac{q}{2}}+\frac{1}{3}\int_{-\infty}^t e^{-\gamma(t-s)}\left(3\alpha_n(s)\right)^{\frac{q}{2}}ds.$$

Using the dominated convergence theorem and the convergence of $\alpha_n(t)$ to 0 as $n\to\infty$, we obtain the convergence of $X^n(t)$ to $X^\infty(t)$ in quadratic mean. Hence $X^n(t)$ converges in distribution to $X^\infty(t)$. But, since the distribution of $X^n(t)$ is the same as that of $X(t+\gamma_n)$, we deduce that, for every $t \in \mathbb{R}$

$$\lim_{n\to\infty} \text{law}(X(t+\alpha_n+\beta_n)) = \text{law}(X^\infty(t)).$$

By analogy and using (4.12) and (4.13), we can easily deduce that

$$\lim_{m\to\infty}\lim_{n\to\infty} \text{law}(X(t+\alpha_m+\beta_n)) = \text{law}(X^\infty(t)).$$

Thus the solution X is almost periodic in one dimensional distributions.
Let us show that $(\|X(t)\|^2)_{t\in\mathbb{R}}$ is uniformly integrable. Using the converse part of Vitali's convergence Theorem, we have that $(\|X_n(t)\|^2)_{n\in\mathbb{N}}$ is uniformly integrable, thus $(\|X(t+\gamma_n)\|^2)$ is uniformly integrable too. As (γ_n'') is arbitrary, this implies that the family $(\|X(t)\|^2)_{t\in\mathbb{R}}$ is uniformly integrable, because, if not, there would exist a sequence (γ_n'') and $t_0 \in \mathbb{R}$ such that no subsequence of $(\|X(t_0+\gamma_n'')\|^2)$ is uniformly integrable. We have thus proved that X has almost periodic one-dimensional 2-distribution.
To prove almost periodicity in 2-distribution of the solution to (4.1), we need a generalization of [28, Proposition 3.1] to the Stepanov context. This allows us to obtain the convergence of the solutions by assuming only the convergence in mean (in Stepanov sense) of the coefficients. Let us mention that a similar result is obtained by Ivo Vrkoč [80], but in another context.

Proposition 4.1.4. *Let $\tau \in \mathbb{R}$. Let $(\xi_n)_{0\le n\le\infty}$ be a sequence of square integrable $\mathbb{H}_2$-valued random variables. Let $(F_n)_{0\le n\le\infty}$ and $(G_n)_{0\le n\le\infty}$ be sequences of mappings from $\mathbb{R}\times\mathbb{H}_2$ to $\mathbb{H}_2$ and $L_2(\mathbb{H}_1,\mathbb{H}_2)$ respectively, which are $\mathbb{S}^2$-bounded and satisfy* (H3) *and* (H4)*, such that the constant M is independent of n and the set of mappings $\{K_n, n\in\mathbb{N}\}$ is $\mathbb{S}^2$-bounded, that is,* $\sup_{n\in\mathbb{N}}\|K_n\|_{\mathbb{S}^2} < +\infty$*. Let X_n denote the solution to*

$$X_n(t) = T(t-\tau)\xi_n + \int_\tau^t T(t-s)F_n\big(s,X_n(s)\big)ds + \int_\tau^t T(t-s)G_n\big(s,X_n(s)\big)dW(s), \quad t\ge\tau.$$

Assume that

$$\lim_{n\to\infty}\|F_n(.,x)-F_\infty(.,x)\|_{\mathbb{S}^2} = 0,\ \lim_{n\to\infty}\|G_n(.,x)-G_\infty(.,x)\|_{\mathbb{S}^2} = 0,$$

for each x in $\mathbb{H}_2$. It follows that
a) There exists a unique mild solution X_∞ to

$$X_\infty(t) = T(t-\tau)\xi_\infty + \int_\tau^t T(t-s)F_\infty\big(s,X_\infty(s)\big)ds + \int_\tau^t T(t-s)G_\infty\big(s,X_\infty(s)\big)dW(s), \quad t\ge\tau. \tag{4.17}$$

b) If $\lim_{n\to\infty}\mathrm{E}\|\xi_n-\xi_\infty\|^2 = 0$*, then, for all $\sigma\ge\tau$,*

$$\lim_{n\to\infty}\mathrm{E}\left(\sup_{\tau\le t\le\sigma}\|X_n(t)-X_\infty(t)\|^2\right) = 0. \tag{4.18}$$

c) If

$$\lim_{n\to\infty} d_{\text{BL}}(\text{law}(\xi_n), \text{law}(\xi_\infty)) = 0,$$

then we have in $C([\tau,\sigma];\mathbb{H}_2)$, for all $\sigma \geq \tau$,

$$\lim_{n\to\infty} d_{\text{BL}}(\text{law}(X_n), \text{law}(X_\infty)) = 0. \tag{4.19}$$

Proof : a) By Proposition 4.1.3, F_∞ and G_∞ satisfy Conditions (H3), (H4), and (H5). We deduce a) as in the first step of the proof of Theorem 4.1.1.

b) For any subsequence (X'_n) of (X_n), we can find by, Proposition 4.1.3, a subsequence (X''_n) of (X'_n) and versions F''_∞ and G''_∞ of F_∞ and G_∞ respectively (i.e., $F''_\infty(t,.) = F_\infty(t,.)$ and $G''_\infty(t,.) = G_\infty(t,.)$ for almost every t), such that the corresponding subsequences (F''_n) and (G''_n) converge pointwise to F''_∞ and G''_∞ respectively. Since the integrals in (4.17) remain unchanged if we replace F_∞ by F''_∞ and G_∞ by G''_∞, we deduce by [28, Proposition 3.1] that

$$\lim_{n\to\infty} \mathrm{E}\left(\sup_{\tau\leq t\leq\sigma} \|X''_n(t) - X_\infty(t)\|^2\right) = 0. \tag{4.20}$$

Thus, for any subsequence (X'_n) of (X_n) we can find a subsequence (X''_n) of (X'_n) such that (4.20) holds, which proves (4.18).

c) Similarly, using [28, Proposition 3.1], we obtain that, for any subsequence (X'_n) of (X_n) we can find a subsequence (X''_n) of (X'_n) such that

$$\lim_{n\to\infty} d_{\text{BL}}(\text{law}(X''_n), \text{law}(X_\infty)) = 0,$$

thus (4.19) holds. □

□

Proof of Theorem 4.1.1 (continued) To prove that X is almost periodic in distribution, we use the same arguments as in [49], using Proposition 4.1.4. □

4.1.2 μ-Pseudo almost periodicity of the solution in 2-distribution

Let μ be a Borel measure on $\mathbb{R}$ satisfying (1.6) and Condition (**H**). Let us start with a useful Lemma :

Lemma 4.1.5. *Let $h \in \mathscr{E}_{\mathbb{S}^q}(\mathbb{R},\mathbb{R},\mu)$, and let $K(.)$ be an $\mathbb{S}^p$-bounded function from $\mathbb{R}$ to $\mathbb{R}^+$. The function*

$$t \mapsto \left(\int_{-\infty}^t e^{-2\delta(t-s)} K^2(s) h^2(s)\, ds\right)^{1/2}$$

is in $\mathscr{E}(\mathbb{R},\mathbb{R},\mu)$.

Proof : Our proof uses the following result of [16, Theorem 3.5], that ensures that $\mathscr{E}_{\mathbb{S}^q}(\mathbb{R},\mathbb{R},\mu)$ is translation invariant. We have, for every $u \in \mathbb{R}$,

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_0^1 |h(t+u+s)|^q \, ds \right)^{\frac{1}{q}} d\mu(t) = 0. \tag{4.21}$$

By Lebesgue's dominated convergence theorem, Hölder's inequality and thanks to Remark 1.2.7, we get

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_{-\infty}^t e^{-2\delta(t-s)} K^2(s) h^2(s)\, ds \right)^{1/2} d\mu(t) \\
&= \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\sum_{k=1}^{+\infty} \int_0^1 e^{-2\delta(k-u)} K^2(t+u-k) h^2(t+u-k)\, du \right)^{1/2} d\mu(t) \\
&\le \lim_{n\to+\infty} \sum_{k=1}^{n} e^{-(k-1)p\delta} \|K\|_{\mathbb{S}^p}^p \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_0^1 h^q(t+u-k)\, du \right)^{\frac{1}{q}} d\mu(t) \\
&\le \lim_{n\to+\infty} \sum_{k=1}^{n} e^{-(k-1)p\delta} \|K\|_{\mathbb{S}^p}^p \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_0^1 h^q(t+u-k)\, du \right)^{\frac{1}{q}} d\mu(t).
\end{aligned}$$

Since the series

$$\sum_{k\ge 1} e^{-(k-1)p\delta} \|K\|_{\mathbb{S}^p}^p \frac{1}{\mu([-r,r])} \int_{[-r,r]} \left(\int_0^1 h^q(t+u-k)\, du \right)^{\frac{1}{q}} d\mu(t)$$

is uniformly convergent with respect to r, the claimed result is a consequence of (4.21). □

□

Before presenting the main result of this subsection, namely, the existence of μ-pseudo almost periodic solution to (4.1), let us introduce the following condition :

(H5)′ $F \in \mathbb{S}^2\mathrm{PAP}(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2,\mu)$ and $G \in \mathbb{S}^2\mathrm{PAP}(\mathbb{R}\times\mathbb{H}_2,\mathrm{L}_2(\mathbb{H}_1,\mathbb{H}_2),\mu)$.

Now, we are able to state the main result of the subsection.

Theorem 4.1.6. *Let the assumptions* (H1) − (H4), (H5)′ *be fulfilled. Let* (F_1,G_1) *and* (F_2,G_2) *be respectively the decompositions of F and G, namely,*

$$\begin{gathered}
F = F_1 + F_2, \quad G = G_1 + G_2, \\
F_1 \in \mathbb{S}^2\mathrm{AP}(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2),\ F_2 \in \mathscr{E}_{\mathbb{S}^2}(\mathbb{R}\times\mathbb{H}_2,\mathbb{H}_2,\mu), \\
G_1 \in \mathbb{S}^2\mathrm{AP}(\mathbb{R}\times\mathbb{H}_2,L_2(\mathbb{H}_1,\mathbb{H}_2)),\ G_2 \in \mathscr{E}_{\mathbb{S}^2}(\mathbb{R}\times\mathbb{H}_2,L_2(\mathbb{H}_1,\mathbb{H}_2),\mu).
\end{gathered}$$

Assume that F_1 and G_1 satisfy the same growth and Lipschitz conditions[4] (H3) *and* (H4)$'$ *as F and G respectively, with same coefficient M and mapping $K(.)$. Assume furthermore that $\theta_{\mathbb{S}} < 1$. Then there exists a unique mild solution X to* (4.1) *in the space* $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ *and X has a.e. continuous trajectories. Moreover, if $\theta'_{\mathbb{S}} < 1$, then X is μ-pseudo almost periodic in* 2*-distribution. More precisely, let $Y \in \mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ be the unique almost periodic in* 2*-distribution mild solution to*

$$dY(t) = AY(t)\,dt + F_1(t, Y(t))\,dt + G_1(t, Y(t))\,dW(t),\ t \in \mathbb{R}. \tag{4.22}$$

Then X has the decomposition

$$X = Y + Z, \quad Z \in \mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2), \mu\big).$$

The proof of this theorem is inspired from [9, Theorem 4.4], which is the analogous result for SDEs with μ-pseudo almost periodic coefficients.

Proof of Theorem 4.1.6 The existence and properties of Y are guaranteed by Theorem 4.1.1.

As in Theorem 4.1.1, the existence and uniqueness of the mild solution X to (4.1) are proved using the classical method of the fixed point theorem for the contractive operator Γ on $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ defined by

$$\Gamma X(t) = \int_{-\infty}^{t} T(t-s) F\big(s, X(s)\big)\,ds + \int_{-\infty}^{t} T(t-s) G\big(s, X(s)\big)\,dW(s).$$

The solution X defined by (4.2) is thus the limit in $\mathrm{CUB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ of a sequence (X_n) with arbitrary X_0 and, for every n, $X_{n+1} = \Gamma(X_n)$. To prove that X is μ-pseudo almost periodic in 2-distribution, we choose a special sequence. Set

$$X_0 = Y,\ X_{n+1} = \Gamma(X_n),\ Z_n = X_n - Y,\ n \in \mathbb{N}.$$

Let us prove by induction that each Z_n is in $\mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2), \mu\big)$. We use some arguments of the proof of [16, Theorem 5.7].

We have, for every $n \in \mathbb{N}$ and every $t \in \mathbb{R}$,

$$Z_{n+1}(t) = \Gamma X_n(t) - Y(t)$$

[4]Let us mention here that in the context of μ-pseudo almost periodicity (resp. μ-pseudo almost automorphy), assuming that F_1 and G_1 satisfy the same growth and Lipschitz conditions as F is not necessary, as already done in [9]. Indeed, assume that F is L-Lipchitz. Set, for each fixed $x, y \in \mathbb{H}_1$, and $t \in \mathbb{R}$, $\hat{F}(t,x,y) := \|F(t,x) - F(t,y)\|$. From the following rewrite of $\hat{F}(t,x,y)$: $\hat{F}(t,x,y) = \hat{F}_1(t,x,y) + \big(\hat{F}(t,x,y) - \hat{F}_1(t,x,y)\big)$, and the μ-ergodicity of the mapping

$$\big[t \mapsto H(t,x,y) := \hat{F}(t,x,y) - \hat{F}_1(t,x,y)\big],$$

we have $\hat{F}(.,x,y) \in \mathrm{PAP}(\mathbb{R}, \mathbb{H}_2, \mu)$. In view of the uniqueness of the previous decomposition (under Condition **(H)**), we deduce that $\big\{\hat{F}_1(t,x,y), t \in \mathbb{R}\big\} \subset \overline{\big\{\hat{F}(t,x,y), t \in \mathbb{R}\big\}}$ (the closure of the range of $\hat{F}$). Consequently, for all $t \in \mathbb{R}$,

$$\hat{F}_1(t,x,y) \le \sup_{t \in \mathbb{R}} \hat{F}(t,x,y) \le L\,\|x-y\|$$

from which we conclude that F_1 is L-Lipschitz.

$$
\begin{aligned}
&= \int_{-\infty}^{t} T(t-s)\big(F(s,X_n(s)) - F(s,Y(s))\big)\, ds \\
&\quad + \int_{-\infty}^{t} T(t-s)\big(G(s,X_n(s)) - G(s,Y(s))\big)\, dW(s) \\
&\quad + \int_{-\infty}^{t} T(t-s)\big(F(s,Y(s)) - F_1(s,Y(s))\big)\, ds \\
&\quad + \int_{-\infty}^{t} T(t-s)\big(G(s,Y(s)) - G_1(s,Y(s))\big)\, dW(s) \\
&= \int_{-\infty}^{t} T(t-s)\big(F(s,X_n(s)) - F(s,Y(s))\big)\, ds \\
&\quad + \int_{-\infty}^{t} T(t-s)\big(G(s,X_n(s)) - G(s,Y(s))\big)\, dW(s) \\
&\quad + \int_{-\infty}^{t} T(t-s)F_2(s,Y(s))\, ds + \int_{-\infty}^{t} T(t-s)G_2(s,Y(s))\, dW(s) \\
&= J_1(t) + J_2(t) + J_3(t).
\end{aligned}
$$

Assume that $Z_n \in \mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$. Since $\mathscr{E}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big) \subset \mathscr{E}_{\mathbb{S}^q}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$, we have by the Lipschitz condition (H4),

$$
\big(\mathrm{E}\,\|F(s,X_n(s)) - F(s,Y(s)\|^2\big)^{1/2} \le K(s)\big(\mathrm{E}\,\|Z_n(s)\|^2\big)^{1/2}, \forall s \in \mathbb{R}.
$$

The same inequality holds for G. Thus, using Hölder's inequality and Lemma 4.1.5, we get

$$
\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]} \Big(\mathrm{E}\,\|J_1(s)\|^2\Big)^{1/2} d\mu(t) \\
&\qquad \le \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\mathrm{E}\left\|\int_{-\infty}^{t} T(t-s)\big(F(s,X_n(s)) - F(s,Y(s))\big)\, ds\right\|^2\right)^{1/2} d\mu(t) \\
&\qquad \le \frac{1}{(\delta)^{1/2}}\frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\int_{-\infty}^{t} e^{-\delta(t-s)} K^2(s)\,\mathrm{E}\,\|Z_n(s)\|^2\, ds\right)^{1/2} d\mu(t) \\
&\qquad \to 0 \text{ when } r \to +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]} \Big(\mathrm{E}\,\|J_2(s)\|^2\Big)^{1/2} d\mu(t) \\
&\qquad \le \frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\mathrm{E}\left\|\int_{-\infty}^{t} T(t-s)\big(G(s,X_n(s)) - G(s,Y(s))\big)\, dW(s)\right\|^2\right)^{1/2} d\mu(t) \\
&\qquad \le (\operatorname{tr} Q)^{1/2}\frac{1}{\mu([-r,r])}\int_{[-r,r]} \left(\int_{-\infty}^{t} e^{-2\delta(t-s)} K^2(s)\,\mathrm{E}\,\|Z_n(s)\|^2\, ds\right)^{1/2} d\mu(t) \\
&\qquad \to 0 \text{ when } r \to +\infty.
\end{aligned}
$$

Hence, $J_1(.)$ and $J_2(.)$ are in $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$. To prove that Z_{n+1} is in $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$, there only remains to show that the process $J_3(.)$ belongs to $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$. As Y is almost periodic in distribution, the family $(Y(u+.))_{u\in\mathbb{R}}$ is uniformly tight in $C_k(\mathbb{R},\mathbb{H}_2)$. In particular, for each $\varepsilon>0$ there exists a compact subset $\mathscr{K}_\varepsilon$ of $C_k(\mathbb{R},\mathbb{H}_2)$ such that, for every $u\in\mathbb{R}$,

$$\mathrm{P}\{Y(u+.)\in\mathscr{K}_\varepsilon\}\geq 1-\varepsilon,$$

which implies that, for every $\varepsilon>0$, there exists a compact subset K_ε of $\mathbb{H}_2$ such that, for every $u,t\in\mathbb{R}$,

$$\mathrm{P}\{(\forall s\in[t,t+1]);\ Y(u+s)\in K_\varepsilon\}\geq 1-\varepsilon.$$

In particular, for $u=0$, we obtain, for every $t\in\mathbb{R}$,

$$\mathrm{P}\{(\forall s\in[t,t+1]);\ Y(s)\in K_\varepsilon\}\geq 1-\varepsilon. \tag{4.23}$$

Let $\Omega_{\varepsilon,t}$ be the measurable subset of Ω on which (4.23) holds. By compactness of K_ε, we can find a finite sequence $y_1,\dots,y_{n(\varepsilon)}$ such that

$$K_\varepsilon\subset\cup_{i=1}^{n(\varepsilon)}B(y_i,\varepsilon),$$

and we get, using (4.23), for every $t\in\mathbb{R}$,

$$\sup_{s\in[t,t+1]}\mathrm{E}\left(\min_{1\leq i\leq n(\varepsilon)}\left(1_{\Omega_{\varepsilon,t}}\|Y(s)-y_i\|^2\right)\right)<\varepsilon. \tag{4.24}$$

Note that $F_2=F-F_1$ and $G_2=G-G_1$ satisfy similar conditions as (H4) and (H3).
We have then by Itô's isometry

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\mathrm{E}\left\|\int_{-\infty}^{t}T(t-s)F_2(s,Y(s))ds+\int_{-\infty}^{t}T(t-s)G_2(s,Y(s))\,dW(s)\right\|^2d\mu(t)\\
\leq&\frac{2\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t}e^{-\delta(t-s)}\,\mathrm{E}\,\|F_2(s,Y(s))\|^2\,dsd\mu(t)\\
&+\frac{2\operatorname{tr}Q}{\mu([-r,r])}\int_{[-r,r]}\int_{-\infty}^{t}e^{-2\delta(t-s)}\,\mathrm{E}\,\|G_2(s,Y(s))\|^2\,dsd\mu(t)\\
=&J_3^1(r)+J_3^2(r).\end{aligned}$$

Let us deal with the term $J_3^1(r)$. We have

$$\begin{aligned}J_3^1(r)\leq&\frac{2\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{+\infty}\int_{t-k}^{t-k+1}e^{-\delta(t-s)}\,\mathrm{E}\,\|F_2(s,Y(s))\|^2\,ds\right)d\mu(t)\\
\leq&\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\sum_{k=1}^{+\infty}e^{-\delta(k-1)}\int_{t-k}^{t-k+1}\mathrm{E}\left(\min_{1\leq i\leq n}\left(\mathbf{1}_{\Omega_{\varepsilon,t}}\|F_2(s,Y(s))-F_2(s,y_i)\|^2\right)\right)dsd\mu(t)\\
&+\max_{1\leq i\leq n}\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t}e^{-\delta(t-s)}\,\mathrm{E}\left(\|F_2(s,y_i)\|^2\right)ds\right)d\mu(t)\\
&+\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^{t}e^{-\delta(t-s)}\,\mathrm{E}\left(\mathbf{1}_{\Omega_{\varepsilon,t}^c}\|F_2(s,Y(s))\|^2\right)ds\right)d\mu(t)\\
=&I_1(r)+I_2(r)+I_3(r).\end{aligned}$$

Using the Lipschitz condition (H4) and the estimation (4.24), we obtain

$$
\begin{aligned}
I_1(r) &\leq \frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{+\infty}e^{-\delta(k-1)}\sup_{t\in\mathbb{R}}\int_t^{t+1}K^2(s)\,\mathrm{E}\Big(\min_{1\leq i\leq n}\big(\mathbf{1}_{\Omega_{\varepsilon,t}}\|Y(s))-y_i\|^2\big)\Big)ds\right)d\mu(t)\\
&\leq \frac{4\delta^{-1}\|K\|^2_{\mathbb{S}^2}}{1-e^{-\delta}}\varepsilon.
\end{aligned}
$$

Thanks to the ergodicity of F_2 and Lebesgue's dominated convergence theorem, one obtains, for any $r>0$,

$$
\begin{aligned}
I_2(r) &= \max_{1\leq i\leq n}\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-\delta(t-s)}\,\mathrm{E}\big(\|F_2(s,y_i)\|^2\big)ds\right)d\mu(t)\\
&\leq \max_{1\leq i\leq n}\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{+\infty}e^{-\delta(k-1)}\int_{t-k}^{t-k+1}\mathrm{E}\big(\|F_2(s,y_i)\|^2\big)ds\right)d\mu(t)\\
&= \max_{1\leq i\leq n}\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{+\infty}e^{-\delta(k-1)}\int_{t-k}^{t-k+1}\mathrm{E}\big(\|F_2(s,y_i)\|^2\big)ds\right)d\mu(t)\\
&= \sum_{k=1}^{+\infty}e^{-\delta(k-1)}\max_{1\leq i\leq n}\frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_0^1\mathrm{E}\big(\|F_2(t-k+s,y_i)\|^2\big)ds\right)d\mu(t).
\end{aligned}
$$

Arguing as in the proof of Lemma 4.1.5, we deduce that $\lim_{r\to\infty}I_2(r)=0$. On the other hand, by Condition (H3) and the uniform integrability of the family $(\|Y(t)\|^2)_{t\in\mathbb{R}}$, we get

$$
\begin{aligned}
I_3(r) &= \frac{4\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-\delta(t-s)}\,\mathrm{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,t}}\|F_2(s,Y(s))\|^2\big)ds\right)d\mu(t)\\
&\leq \frac{4M\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\int_{-\infty}^t e^{-\delta(t-s)}\,\mathrm{E}\big(\mathbf{1}_{\Omega^c_{(\varepsilon,t)}}(1+\|Y(s))\|)^2\big)ds\right)d\mu(t)\\
&\leq \frac{8M\delta^{-1}}{\mu([-r,r])}\int_{[-r,r]}\left(\sum_{k=1}^{+\infty}e^{-\delta(k-1)}\left(\mathrm{P}\big(\Omega^c_{\varepsilon,t}\big)+\int_{t-k}^{t-k+1}\mathrm{E}\big(\mathbf{1}_{\Omega^c_{\varepsilon,t}}\|Y(s))\|^2\big)ds\right)\right)d\mu(t)\\
&\leq \frac{8M\delta^{-1}\varepsilon}{1-e^{-\delta}}.
\end{aligned}
$$

As ε is arbitrary, we deduce from the previous estimations on $I_1(r), I_2(r)$ and $I_3(r)$, that $\lim_{r\to\infty}J_3^1(r)=0$. In the same way, we can estimate the term $J_3^2(r)$. We then easily see that $\lim_{r\to\infty}J_3^2(r)=0$. Consequently, $Z_{n+1}\in\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$.
So we have shown that the sequence (Z_n) lies in $\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$.
Now, the sequence (X_n) converges to X in $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$, thus (Z_n) converges to $Z:=X-Y$ in $\mathrm{CUB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$. Let $\varepsilon>0$, and let $n\in\mathbb{N}$ be such that

$$
\sup_{t\in\mathbb{R}}\big(\mathrm{E}\|Z(t)-Z_n(t)\|^2\big)^{1/2}\leq\varepsilon.
$$

We have

$$
\frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\|Z(t)\|^2\big)^{1/2}d\mu(t)\leq\frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\|Z(t)-Z_n(t)\|^2\big)^{1/2}d\mu(t)
$$

$$+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\,\|Z_n(t)\|^2\big)^{1/2}d\mu(t)$$
$$\leq\varepsilon+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\big(\mathrm{E}\,\|Z_n(t)\|^2\big)^{1/2}d\mu(t).$$

which proves that $Z\in\mathscr{E}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2),\mu\big)$, since $\varepsilon>0$ is arbitrary. □

4.2 Comments and concluding remarks

When the function g in (4.1) is equal to zero, we retrieve a semilinear (deterministic) differential equation in the Banach space $\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)$:

$$u'(t)=Au(t)+f(t,u(t)),\, t\in\mathbb{R}. \tag{4.25}$$

An extensive literature (see for e.g. [39,38,40,59,53,89]), is devoted to the problem of the existence and uniqueness of a bounded (μ-pseudo) almost periodic mild solution to (4.25) in a Banach space $\mathbb{X}$. The adopted approach is based on superposition theorems in the Banach space $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$ (or $\mathbb{S}^p\mathrm{PAP}(\mathbb{R},\mathbb{X},\mu)$) combined with the Banach's fixed-point principle, applied to the nonlinear operator

$$(\Gamma u)(t)=\int_{-\infty}^{t}T(t-s)f(s,u(s))ds.$$

To our knowledge, all existing results use the fact that Γ maps $\mathrm{AP}(\mathbb{R},\mathbb{X})$ into itself, but Γ does not map $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$ into $\mathrm{AP}(\mathbb{R},\mathbb{X})$ nor into $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$ (see in particular [40,59, 53,89]). The proposed proofs may be summarized as follows : if $u\in\mathrm{AP}(\mathbb{R},\mathbb{X})$, then u satisfies the compactness condition (Com) of Subsection **??**, and $u\in\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$. From the existing superposition theorems (see e.g. [40, Theorem 2.1]) combined with Condition (Lip), it follows that $F(.):=f(.,u(.))\in\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$, and then $(\Gamma u)(.)=\int_{-\infty}^{.}T(.-s)F(s)ds\in\mathrm{AP}(\mathbb{R},\mathbb{X})$. This obviously shows the existence (and uniqueness) of an almost periodic mild solution to (4.25), but it does not exclude the possibility of existence of a purely Stepanov almost periodic solution. The main difficulty in showing the non-existence of a purely Stepanov almost periodic bounded solution with the tools used in the literature (see for example [40]), arises from the imposed compactness condition (Com) in the superposition theorem of Stepanov almost periodic functions, which seems strong enough in $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$. Thanks to Theorem 2.2.2, it is easy to see that under Condition (Lip), the operator Γ maps $\mathbb{S}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$ into $\mathrm{AP}(\mathbb{R},\mathbb{X})$. This shows that the obtained bounded solution cannot be purely Stepanov almost periodic.
Now, we focus on the following problem : can we expect to have a similar conclusion if we replace the assumption that f is Stepanov almost periodic by the assumption that f is almost periodic in Lebesgue measure ? Obviously, the answer depends on the Bohl-Bohr-Amerio Theorem, see e.g. [52, p. 80], for functions which are almost periodic in Lebesgue measure. What we need is to solve first another problem, namely, does the boundedness of the indefinite integral of a function which is almost periodic in Lebesgue measure imply its Bohr almost-periodicity ? The answer to this question is negative as shown by the following example :

Example 4.2.1. It is well-known that the Levitan function $H : \mathbb{R} \to \mathbb{R}$, defined by

$$H(t) = \sin\left(\frac{1}{g(t)}\right),$$

where

$$g(t) = 2 + \cos(t) + \cos(\sqrt{2}t)$$

is a bounded $\mathbb{S}^p$-almost periodic function but not Bohr-almost periodic (see e.g. [52]). Let's denote by h its derivative. We have

$$h(t) = \cos\left(\frac{1}{g(t)}\right)\left(\frac{\sin(t) + \sqrt{2}\sin(\sqrt{2}t)}{g^2(t)}\right). \tag{4.26}$$

By the Bohl-Bohr Theorem, the derivative function h cannot be Stepanov almost periodic (see [5]), and consequently, the function $1/g$ cannot be Stepanov almost periodic. But one can easily observe by taking into account 2. in Remark 1.2.3, that h is in $\mathscr{S}_{\varkappa}(\mathbb{R})$ as a product of $\mathscr{S}_{\varkappa}(\mathbb{R})$-functions.

In order to answer the first problem, we give, in the following, a simple affine scalar equation with purely almost periodic in Lebesque measure coefficient. We see that the unique bounded solution is not Bohr-almost periodic (but it is purely Stepanov almost periodic).

Example 4.2.2. Consider the affine differential equation

$$x'(t) = -x(t) + h(t), t \in \mathbb{R}, \tag{4.27}$$

where h is given by (4.26). The unique bounded solution to (4.27) is given by :

$$x(t) = \int_{-\infty}^{t} e^{(s-t)} h(s) ds = \sin\left(\frac{1}{g(t)}\right) + \int_{-\infty}^{t} e^{(s-t)} \sin\left(\frac{1}{g(s)}\right) ds.$$

The boundedness of x follows from

$$|x(t)| \le \left|\sin\left(\frac{1}{g(t)}\right)\right| + \int_{-\infty}^{t} e^{(s-t)} \left|\sin\left(\frac{1}{g(s)}\right)\right| ds \le 2, \ \forall t \in \mathbb{R}.$$

But x is not Bohr-almost periodic, as it is the sum of the purely Stepanov almost periodic function $H(t) = \sin\left(\frac{1}{g(t)}\right)$ and a Bohr almost periodic function.

To move towards the consequences obtained by Andres and Pennequin [5, Consequence 1, p. 1667 and Consequence 4, p. 1679], this example shows (in addition) that one can obtain the existence and the uniqueness of a bounded purely Stepanov almost periodic solution when the coefficients are purely almost periodic in Lebesgue measure.
This simple result can open new directions about the problem of existence of purely Stepanov almost periodic solutions, in both stochastic and deterministic cases. □

Bibliography

Bibliographie

[1] Syed Abbas. A note on Weyl pseudo almost automorphic functions and their properties. *Math. Sci. (Springer)*, 6 :Art. 29, 5, 2012.

[2] L. Amerio and G. Prouse. *Almost-periodic functions and functional equations. (The University Series in Higher Mathematics.)*. 1971. Published : New York etc. : Van Nostrand Reinhold Company VIII, 184 p. (1971).

[3] J. Andres, A. M. Bersani, and R. F. Grande. Hierarchy of almost-periodic function spaces. *Rend. Mat. Appl., VII. Ser.*, 26(2) :121–188, 2006.

[4] Jan Andres, Alberto M. Bersani, and Krzysztof Leśniak. On some almost-periodicity problems in various metrics. *Acta Appl. Math.*, 65(1-3) :35–57, 2001.

[5] Jan Andres and Denis Pennequin. On Stepanov almost-periodic oscillations and their discretizations. *J. Difference Equ. Appl.*, 18(10) :1665–1682, 2012.

[6] Jan Andres and Denis Pennequin. On the nonexistence of purely Stepanov almost-periodic solutions of ordinary differential equations. *Proc. Am. Math. Soc.*, 140(8) :2825–2834, 2012.

[7] Ludwig Arnold and Constantin Tudor. Stationary and almost periodic solutions of almost periodic affine stochastic differential equations. *Stochastics Stochastics Rep.*, 64(3-4) :177–193, 1998.

[8] A. A. Pankov (auth.). *Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations*. Mathematics and its Applications 55. Springer Netherlands, 1 edition, 1990.

[9] Fazia Bedouhene, Nouredine Challali, Omar Mellah, Paul Raynaud de Fitte, and Mannal Smaali. Almost automorphy and various extensions for stochastic processes. *J. Math. Anal. Appl.*, 429(2) :1113–1152, 2015.

[10] Fazia Bedouhene, Omar Mellah, and Paul Raynaud de Fitte. Bochner-almost periodicity for stochastic processes. *Stoch. Anal. Appl.*, 30(2) :322–342, 2012.

[11] Abram Samoilovitch Besicovitch. *Almost periodic functions*. Dover Publications, 1954.

[12] Paul H. Bezandry and Toka Diagana. Existence of almost periodic solutions to some stochastic differential equations. *Appl. Anal.*, 86(7) :819–827, 2007.

[13] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics : Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[14] J. Blot, G. M. Mophou, G. M. N'Guérékata, and D. Pennequin. Weighted pseudo almost automorphic functions and applications to abstract differential equations. *Nonlinear Anal.*, 71(3-4) :903–909, 2009.

[15] Joel Blot and Philippe Cieutat. Completeness of sums of subspaces of bounded functions and applications. *Commun. Math. Anal.*, 19(2) :43–61, 2016.

[16] Joël Blot, Philippe Cieutat, and Khalil Ezzinbi. Measure theory and pseudo almost automorphic functions : new developments and applications. *Nonlinear Anal.*, 75(4) :2426 – 2447, 2012.

[17] Joël Blot, Philippe Cieutat, and Khalil Ezzinbi. New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications. *Appl. Anal.*, 92(3) :493–526, 2013.

[18] Joël Blot, Philippe Cieutat, Gaston M. N'Guérékata, and Denis Pennequin. Superposition operators between various almost periodic function spaces and applications. *Commun. Math. Anal.*, 6(1) :42–70, 2009.

[19] S. Bochner. Abstrakte Fastperiodische Funktionen. *Acta Math.*, 61(1) :149–184, 1933.

[20] S. Bochner. A new approach to almost periodicity. *Proc. Nat. Acad. Sci. U.S.A.*, 48 :2039–2043, 1962.

[21] S. Bochner. Continuous mappings of almost automorphic and almost periodic functions. *Proc. Nat. Acad. Sci. U.S.A.*, 52 :907–910, 1964.

[22] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chapitres 1 à 4*. Hermann, Paris, 1971.

[23] Valentina Casarino. Almost automorphic groups and semigroups. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 24 :219–235, 2000.

[24] Gh. Cenuşă and I. Săcuiu. Some properties of random functions almost periodic in probability. *Rev.Roumaine Math. Pures Appl.*, 25(9) :1317–1325, 1980.

[25] Yong-Kui Chang, Zhi-Han Zhao, Gaston M. N'Guerekata, and Ruyun Ma. Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations. *Nonlinear Anal., Real World Appl.*, 12(2) :1130–1139, 2011.

[26] C. Corduneanu. *Almost periodic functions*. Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1968. With the collaboration of N. Gheorghiu and V. Barbu, Translated from the Romanian by Gitta Bernstein and Eugene Tomer, Interscience Tracts in Pure and Applied Mathematics, No. 22.

[27] C. Corduneanu. *Almost periodic functions. With the collaboration of N. Gheorghiu and V. Barbu. Transl. from the Romanian by Gitta Berstein and Eugene Tomer. 2nd Engl. ed.* New York : Chelsea Publishing Company, 2nd engl. ed. edition, 1989.

[28] G. Da Prato and C. Tudor. Periodic and almost periodic solutions for semilinear stochastic equations. *Stochastic Anal. Appl.*, 13(1) :13–33, 1995.

[29] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[30] L. I. Danilov. Measure-valued almost periodic functions. *Math. Notes*, 61(1) :48–57, 1997.

[31] L. I. Danilov. Measure-valued almost periodic functions and almost periodic selections of multivalued mappings. *Mat. Sb.*, 188(10) :3–24, 1997.

[32] L. I. Danilov. On the uniform approximation of a function that is almost periodic in the sense of Stepanov. *Izv. Vyssh. Uchebn. Zaved. Mat.*, 5 :10–18, 1998.

[33] L. I. Danilov. Almost periodic measure-valued functions. *Sb. Math.*, 191(12) :1773–1796, 2000.

[34] Toka Diagana. Weighted pseudo almost periodic functions and applications. *C. R. Math. Acad. Sci. Paris*, 343(10) :643–646, 2006.

[35] Toka Diagana. *Pseudo almost periodic functions in Banach spaces*. Nova Science Publishers, Inc., New York, 2007.

[36] Toka Diagana. Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 69(12) :4277–4285, 2008.

[37] Toka Diagana. Weighted pseudo-almost periodic solutions to some differential equations. *Nonlinear Anal.*, 68(8) :2250–2260, 2008.

[38] Toka Diagana. Existence of pseudo-almost automorphic solutions to some abstract differential equations with $\mathbf{S}^p$-pseudo-almost automorphic coefficients. *Nonlinear Anal.*, 70(11) :3781–3790, 2009.

[39] Toka Diagana. Evolution equations in generalized Stepanov-like pseudo almost automorphic spaces. *Electron. J. Differential Equations*, pages No. 49, 19, 2012.

[40] Hui-Sheng Ding, Wei Long, and Gaston M. N'Guérékata. Almost periodic solutions to abstract semilinear evolution equations with Stepanov almost periodic coefficients. *J. Comput. Anal. Appl.*, 13(2) :231–242, 2011.

[41] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[42] Zhenbin Fan, Jin Liang, and Ti-Jun Xiao. On Stepanov-like (pseudo) almost automorphic functions. *Nonlinear Analysis : Theory, Methods & Applications*, 74(8) :2853 – 2861, 2011.

[43] A. M. Fink. *Almost periodic differential equations*. Lecture Notes in Mathematics, Vol. 377. Springer-Verlag, Berlin, 1974.

[44] M. Fu and Z. Liu. Square-mean almost automorphic solutions for some stochastic differential equations. *Proceedings of the American Mathematical Society*, 138(10) :3689–3701, 2010.

[45] Miaomiao Fu. Almost automorphic solutions for nonautonomous stochastic differential equations. *Journal of Mathematical Analysis and Applications*, 393(1) :231 – 238, 2012.

[46] Miaomiao Fu and Feng Chen. Almost automorphic solutions for some stochastic differential equations. *Nonlinear Anal.*, 80 :66–75, 2013.

[47] A. Halanay. Periodic and almost periodic solutions to affine stochastic systems. In *Proceedings of the Eleventh International Conference on Nonlinear Oscillations (Budapest, 1987)*, pages 94–101, Budapest, 1987. János Bolyai Math. Soc.

[48] Zuosheng Hu and Angelo B. Mingarelli. Bochner's theorem and Stepanov almost periodic functions. *Ann. Mat. Pura Appl. (4)*, 187(4) :719–736, 2008.

[49] Mikhail Kamenskii, Omar Mellah, and Paul Raynaud de Fitte. Weak averaging of semilinear stochastic differential equations with almost periodic coefficients. *J. Math. Anal. Appl.*, 427(1) :336–364, 2015.

[50] J. Komlós. A generalization of a problem of Steinhaus. *Acta Mathematica Hungarica*, 18(1-2) :217–229, 1967.

[51] B. M. Levitan. *Almost periodic functions*. Moskva : Gosudarstv. Izdat. Tehn.-Teor. Lit., 1953.

[52] B. M. Levitan and V. V. Zhikov. *Almost periodic functions and differential equations*. CUP Archive, 1982.

[53] Hong-Xu Li and Li-Li Zhang. Stepanov-like pseudo-almost periodicity and semilinear differential equations with uniform continuity. *Result. Math.*, 59(1-2) :43–61, 2011.

[54] Jin Liang, Ti-Jun Xiao, and Jun Zhang. Decomposition of weighted pseudo-almost periodic functions. *Nonlinear Anal.*, 73(10) :3456–3461, 2010.

[55] Jin Liang, Jun Zhang, and Ti-Jun Xiao. Composition of pseudo almost automorphic and asymptotically almost automorphic functions. *J. Math. Anal. Appl.*, 340(2) :1493–1499, 2008.

[56] Georg Lindgren. Lectures on stationary stochastic processes. Technical report, Lund University, Lund, Sweden, 2006. A course for PHD students in Mathematical Statistics and other fields available at http ://www.maths.lth.se/matstat/staff/georg/Publications/lecture2006.pdf.

[57] Zhenxin Liu and Kai Sun. Almost automorphic solutions for stochastic differential equations driven by Lévy noise. *J. Funct. Anal.*, 266(3) :1115–1149, 2014.

[58] Carlos Lizama and Gaston M. N'Guérékata. Bounded mild solutions for semilinear integro differential equations in Banach spaces. *Integral Equations Operator Theory*, 68(2) :207–227, 2010.

[59] Wei Long and Hui-Sheng Ding. Composition theorems of Stepanov almost periodic functions and Stepanov-like pseudo-almost periodic functions. *Adv. Difference Equ.*, 2011 :12, 2011.

[60] Omar Mellah and Paul Raynaud de Fitte. Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients. *Electron. J. Differential Equations*, pages No. 91, 7, 2013.

[61] T. Morozan and C. Tudor. Almost periodic solutions of affine Itô equations. *Stochastic Anal. Appl.*, 7(4) :451–474, 1989.

[62] Gaston M. N'guérékata. Comments on almost automorphic and almost periodic functions in Banach spaces. *Far East J. Math. Sci. (FJMS)*, 17(3) :337–344, 2005.

[63] Gaston M. N'Guérékata. *Topics in almost automorphy*. Springer-Verlag, New York, 2005.

[64] Gaston M. N'Guérékata and Alexander Pankov. Stepanov-like almost automorphic functions and monotone evolution equations. *Nonlinear Anal.*, 68(9) :2658–2667, 2008.

[65] G.M. N'Guérékata. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*. Springer, 2001.

[66] Octav Onicescu and Vasile I. Istrătescu. Approximation theorems for random functions. *Rend. Mat. (6)*, 8 :65–81, 1975. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday.

[67] Justyna Signerska-Rynkowska Piotr Kasprzak, Adam Nawrocki. Integrate-and-fire models with an almost periodic input function. *ArXiv e-prints*, 1610.04434v1, October 2016.

[68] A. M. Precupanu. On the almost periodic functions in probability. *Rend. Mat. (7)*, 2(3) :613–626, 1982.

[69] Aribindi Satyanarayan Rao. On the almost periodic solution of a second-order infinitesimal generator differential equation. *Bull. Calcutta Math. Soc.*, 91(5) :391–396, 1999.

[70] Aribindi Satyanarayan Rao. On a higher-order evolution equation with a Stepanov-bounded solution. *Int. J. Math. Math. Sci.*, 2004(69-72) :3959–3964, 2004.

[71] W. Shen and Y. Yi. *Almost Automorphic and Almost Periodic Dynamics in Skew-product Semiflows*. Number vol. 647 in Almost automorphic and almost periodic dynamics in skew-product semiflows. American Mathematical Soc., 1998.

[72] E. Slutsky. Sur les fonctions aléatoires presque périodiques et sur la décomposition des fonctions aléatoires en composantes stationnaires. pages 33–55, 1938.

[73] W. Stepanoff. Über einige verallgemeinerungen der fast periodischen funktionen. *Mathematische Annalen*, 95(1) :473–498, 1926.

[74] Stanisław Stoiński. Almost periodic functions in the Lebesgue measure. *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, 34 :189–198, 1994.

[75] Stanisław Stoiński. On compactness of almost periodic functions in the Lebesgue measure. *Fasc. Math.*, 30 :171–175, 1999.

[76] C. Tudor. Almost periodic solutions of affine stochastic evolution equations. *Stochastics Stochastics Rep.*, 38(4) :251–266, 1992.

[77] C. Tudor. Almost periodic stochastic processes. In *Qualitative problems for differential equations and control theory*, pages 289–300. World Sci. Publ., River Edge, NJ, 1995.

[78] C. A. Tudor and M. Tudor. Pseudo almost periodic solutions of some stochastic differential equations. *Math. Rep. (Bucur.)*, 1(51)(2) :305–314, 1999.

[79] W. A. Veech. Almost automorphic functions on groups. *Amer. J. Math.*, 87 :719–751, 1965.

[80] Ivo Vrkoč. Weak averaging of stochastic evolution equations. *Math. Bohem.*, 120(1) :91–111, 1995.

[81] Ward Whitt. Weak convergence of probability measures on the function space $C[0,\infty)$. *Ann. Math. Statist.*, 41 :939–944, 1970.

[82] Zhinan Xia and Meng Fan. Weighted Stepanov-like pseudo almost automorphy and applications. *Nonlinear Anal.*, 75(4) :2378–2397, 2012.

[83] Ti-Jun Xiao, Jin Liang, and Jun Zhang. Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces. *Semigroup Forum*, 76(3) :518–524, 2008.

[84] Chuan Yi Zhang. Integration of vector-valued pseudo-almost periodic functions. *Proc. Amer. Math. Soc.*, 121(1) :167–174, 1994.

[85] Chuan Yi Zhang. Pseudo-almost-periodic solutions of some differential equations. *J. Math. Anal. Appl.*, 181(1) :62–76, 1994.

[86] Chuan Yi Zhang. Pseudo almost periodic solutions of some differential equations. II. *J. Math. Anal. Appl.*, 192(2) :543–561, 1995.

[87] Chuanyi Zhang. *Almost periodic type functions and ergodicity*. Kluwer Academic Publishers, Dordrecht, 2003.

[88] Rui Zhang, Yong-Kui Chang, and G. M. N'Guérékata. New composition theorems of Stepanov-like weighted pseudo almost automorphic functions and applications to nonautonomous evolution equations. *Nonlinear Anal. Real World Appl.*, 13(6) :2866–2879, 2012.

[89] Zhi-Han Zhao, Yong-Kui Chang, and Gaston M. N'Guérékata. A new composition theorem for S^p-weighted pseudo almost periodic functions and applications to semilinear differential equations. *Opusc. Math.*, 31(3) :457–474, 2011.

Résumé

Dans le cadre de cette these, nous avons comparé différents concepts et de pseudo presque automorphie (et ses variantes) pour les processus stochastiques : en probabilité, en moyenne quadratique et en distribution dans divers sens. Nous avons montré par un contre-exemple que la (pseudo) presque automorphie en moyenne quadratique est une propriété qui est trop forte pour les équations différentielles stochastiques. Enfin, nous avons considéré deux EDS semi linéaires, une avec des coefficients presque automorphes et la seconde avec des coefficients pseudo presque automorphes , et nous avons montré l' existence et l'unicité d'une solution mild qui est presque automorphe en distribution dans le premier cas, et pseudo presque automorphe en distribution dans le second cas.

En ce qui concerne la classe des fonctions Stepanov presque périodiques, nous avons déjà énoncé et démontré un nouveau théorème de superposition de fonctions Stepanov presque périodiques. En suite, on s'est attelé à démontrer que les EDS considérées dans notre première étude mais avec des coefficients Stepanov (pseudo) presque périodiques admettent des solutions mild bornées en moyenne quadratique et sont Bohr (pseudo) presque périodiques en distribution.

Abstract

In this thesis, we compared different concepts and pseudo almost automorphy (and its variants) for stochastic processes : probability, quadratic mean and distribution in various directions. We have shown by a counterexample that the (pseudo) almost automorphic in quadratic mean is a property which is too strong for the stochastic differential equations. Finally, we considered two semi - linear EDSs, the first equation with almost automorphic coefficients and the second one with pseudo - almost automorphic coefficients, and we have shown the existence and uniqueness of a mild solution which is almost automorphic in distribution In the first case, and pseudo almost automorphic in distribution in the second case.

As for the class of almost periodic Stepanov functions, we have already stated and demonstrated a new theorem of superposition of almost periodic Stepanov functions. We then proceeded to demonstrate that the EDS considered in our first study but with almost periodic Stepanov (pseudo) coefficients admit solutions mild bounded on quadratic mean and are Bohr (pseudo) almost periodic in distribution.