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Improvement combined with (analysis, CSP, Arithmetic analysis and Interval) of Simulations for efficient combination of two lower bound functions in (univariate,multivariate) global optimization and generalization

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Résumé : Univariate global optimization problems attract attention of researchers.Several methods [23] have been studied in the literature for univariate global optimization problems . Optimization in R presents the same difficulty as in R^n. Many algorithms are directed in this direction. For cutting methods in Global optimization or Optimsation gradient method in general . In this work, we propose to improve: The article submitted: (Simulations for efficient combination of two lower bound functions in univariate global optimization. AIP Conference Proceedings 1863, 250004 (2017) ; https ://doi.org/10.1063/1.4992412, (2017).) In this context too, we will accelerate the speed of the Algorithm for better complexity with technics (CSP, Arithmetic analysis and Interval and another). It should be noted that, we have made conclusive simulations in this direction .

Mots-Clefs : Global optimization, αBB method, quadratic lower bound function, Branch and Bound, pruning method.

Classification MSC2010 : MSC2010 : 90-08 / 90C26

1 Introduction

Global optimization problems are reputed to be the most difficult problems to solve in the field of optimization, R.O., applied mathematics, optimal control ..... Generally, the applied methods do not provide the exact optimum because in general, these methods give local optimums.This is why we opt for global optimization methods.But the problem does not stop here, because it is important to highlight the multiplicities of the solutions, in addition there is also a problem of algorithmic complexity to be improved every time by using the test problems.Univariate global optimization problems attract attention of researchers.Several methods [**?**] have been studied in the literature for univariate global optimization problems .Optimization in R presents the same difficulty as in R^n. Many algorithms are directed in this direction. For cutting methods in Global optimization or Optimsation gradient method in general . In this work, we propose to improve: The article submitted: (Simulations for efficient combination of two lower bound functions in univariate global optimization. AIP Conference Proceedings 1863, 250004 (2017) ; https ://doi.org/10.1063/1.4992412, (2017).) In this context too, we will accelerate the speed of the Algorithm for better complexity with technics (CSP, Arithmetic analysis and Interval and another). It should be noted that, we have made conclusive simulations in this direction .
Our work is built on 03 steps
The first step is to solve any optimization problem in dimension 1.
The second step: it is to solve a problem of any optimation in dimension $n > 1$, with separable

variables, which will introduce us to the third stage (step).
The third step is to solve any optimization problem in dimension $n > 1$ using the results of the first two steps, the theoretical results concerning this step and our main personal contribution.We will test this on for example 10 problems test . in order to test the effectiveness of our work. Our goal too :: 1 / Improve the performance of the Algorithm in ([?]). 2 / simplify this algorithm. 3 / Apply the new Algorithm for better complexity on the 10 test problems and other test problems.

We consider the following problem

$$(P)\begin{cases} \min f(x) \\ x \in [x^0, x^1] \subset R \end{cases}$$

Main Improvements (Main Contributions)

Improvements are (Types C.S.P)
1 / Extra rapid convexity test (test A_1)
2 / Computation of bounds, to inhibit intervals by anlyse intervals or affine arithmetic (test A_2)
3 / The derivative and its bound, to inhibit intervals by anlyse intervals or affine arithmetic.(test A_3)
4/ if Smooth Form involved Direct Execution.

These 04 procedures will be integrated in the process of the Algorithm, to accelerate the speed of convergence towards the optimal solution.

global optimization with a single variable is not easy because; the functions must be see and especially their forms in a general way. In real cases these functions do not offer facilities for studying them. Many research and methods in global optimization can not bypass global optimization with a single variable, in other words: depend on the study to a variable to find the optimal solution.
with $f(x)$ a non-convex C^2-continuous function on the interval $[x^0, x^1]$ of R.
Univariate global optimization problems attract attention of researchers not only because they arise in many real-life applications but also the methods for these problems are useful for the extension for the multivariate case or by reducing the multidimensional case to the univariate case. One class of deterministic approaches, which called lower bounding method, emerged from the natural strategy to find a global minimum for sure. The efficiency of a method is in the construction of tight lower bound and to discard a large regions which do not contain the global minimum as quickly as possible.

In order to solve the global optimization problem, many envelope methods have been proposed (see [21] and references therein). Several methods have been studied in the literature for univariate global optimization problems, among them we can cite the classical αBB method developed in [19], another method using a quadratic lower bound is developed in [23] for univariate case. The latter is generalized to multivariate case in [25]. In [21], tight convex lower bound for univariate C^2-continuous functions are proposed by using a piecewise quadratic lower bound obtained by αBB method which allows to find convex envelope in finite number of subdivisions. In [24], a branch and prune algorithm is proposed, the pruning step(outer and inner) consists in solving linear equation, the linear bounding function is obtained by interval analysis.

2 Background

2.1 Lower bound function in αBB method [19]

The lower bound function in αBB method on the interval $[x^0, x^1]$ is given by :

$$LB_\alpha(x) = f(x) - \frac{K_\alpha}{2}(x - x^0)(x^1 - x)$$

with $K_\alpha \geq \max\{0, -f''(x)\}, \forall x \in [x^0, x^1]$. The main properties of this lower bound function are:

1. It is convex (i.e. $LB''_\alpha(x) = f''(x) + K_\alpha \geq 0, \forall x \in [x^0, x^1]$).
2. It coincides with the function $f(x)$ at the endpoints of the interval $[x^0, x^1]$ (i.e. by construction of $(LB_\alpha(x))$.
3. It is a lower bound function (i.e. $f(x) - LB_\alpha(x) = \frac{K_\alpha}{2}(x - x^0)(x^1 - x) \geq 0, \forall x \in [x^0, x^1]$).

For more details one see [19].

2.2 Quadratic lower bound function [23]

The quadratic lower bound developed in [23] on the interval $[x^0, x^1]$ is given by :

$$LB_{LO}(x) = f(x^0)\frac{x^1 - x}{x^1 - x^0} + f(x^1)\frac{x - x^0}{x^1 - x^0} - \frac{K}{2}(x - x^0)(x^1 - x)$$

with $K \geq |f''(x)|, \forall x \in [x_0, x_1]$. The main properties of this lower bound function are:

1. It is convex (i.e. $LB''_{LO}(x) = K \geq 0$).
2. It coincides with the function $f(x)$ at the endpoints of the interval $[x^0, x^1]$ (i.e. by construction of $LB_{LO}(x)$).
3. It is a lower bound function (i.e. $(f(x) - LB_{LO}(x))'' = f''(x) - K \leq 0, \forall x \in [x^0, x^1]$.) which implies that $(f(x) - LB_{LO}(x))$ is concave, it vanishes at the endpoints of $[x^0, x^1]$ then $f(x) \geq LB_{LO}(x), \forall x \in [x^0, x^1]$.

Details of the main results

The optimization of univariate functions presents the same difficulties as the functions to multivariate. We find this for example in gradient methods and eigen value calculations .

1/ (test A_1)
The properties of functions exploited to produce formulations that can express convexity, concavity and invexity.
2/ (test A_2)
We use in this context, the properties of the functions, lower bound in different forms.
3/ (test A_3)
In another context, the properties of functions, derivability and differentiability are used.
Remark
Property reformulations of exploited functions to produce simple forms that can be effectively used.

3 Branch and Bound Algorithm and its convergence

The Branch and Bound algorithm is an efficient algorithm. he gave a lot of experimental evidence. This algorithm exists on several variants. We use one of its variants in this document. Many works in global optimization, notably in DC / DCA, global reverse-convex optimization, global optimization type reformulation and overall multi-objective stochastic blur optimization use Branch and bound variants.
Method based on Branch-and-bound (BB) is one of the most popular deterministic global optimization frameworks. It consists on subdividing the solution space into smaller regions where the upper and lower bounds to the objective function value are computed. According to these bounds, each region is explored or fathomed out of the built Branch and Bound tree. Global solution is then obtained once the current best upper bound (UB) value is close to current best lower bound (LB) value within a specified tolerance ε. In this section, we introduce the algorithm for finding the global solution of problem (P) and we show its convergence.

Algorithm Branch and Bound (BB)

Step 1 : Initialization

a0) if Smooth Form involved Direct Execution.

a) Let ε be a given small number and let $[a_0, b_0]$ the initial interval

b) Compute $K_\alpha^0 = \max\{0, sup_{x\in[a_0,b_0]}(-f''(x))\}$ and $K_q^0 = \max\{0, sup_{x\in[a_0,b_0]} f''(x)\}$

b1) Test C.S.P A_1

b2) Test C.S.P A_2

b3) Test C.S.P A_3

c) Apply Convex/concave test

d) Apply the pruning test in order to reduce and update the searching interval

e) Set $k := 0; T^0 = [a_0, b_0]; M := T^0$

f) Compute $LB_\alpha^0(x)$ and $LB_q^0(x)$ on T^0, and solve the convex program to obtain an optimal solution z^0 and s_0^*.

$$\min\left\{z : LB_\alpha^0(x) \le z, LB_q^0(x) \le z, z \in R, x \in T^0\right\} \tag{1}$$

g) Set $UB_0 := \min\left\{f(a_0), f(b_0), f(s_0^*)\right\} = f(\overline{s}^0)$, $LB_0 = LB(T^0) := z^0$.

h) **If** $UB_0 - LB_0 \le \varepsilon$ **then print** $\overline{\mathbf{s}}^0$ as an ε-optimal solution; **EXIT** the algorithm.
else Set $M \leftarrow \{T^0\}$, $k \leftarrow 1$

Step 2 : Iteration

a) Selection step

- Select $T^k = [a_k, b_k] \in M$, the interval such that $LB_k = \min LB(T^k)$

b) Bisection step

- Bisect T^k into two sub-rectangles $T_1^k = [a_k^1, b_k^1], T_2^k = [a_k^2, b_k^2]$ by w-subdivision procedure via $\mathbf{s}*\ ^k$

c) Computing step

- **For** $i = 1, 2$ **do**
  1. Compute K_α^{ki} and K_q^{ki} on the interval T_i^k
  2. Convex test : if $K_\alpha^{ki} = 0$ then update $LB(T_i^k)$ and $UB(T_i^k)$ and go to **step d**
  3. Concave test: if $K_q^{ki} = 0$ then update $LB(T_i^k)$ and $UB(T_i^k)$ and go to **step d**

  31) Test C.S.P A_1 on the interval T_i^k

32) Test C.S.P A_2 on the interval T_i^k

33) Test C.S.P A_3 on the interval T_i^k

4. Pruning test : Compute LB_q^{ki} and solve $LB_q^{ki} = UB_k$ to reduce the searching interval $[a_k^i, b_k^i]$
5. Compute $LB_\alpha^{ki}(x)$. Let z^{ki} and s_{ki}^* be the solution of the convex problem

$$\min\left\{z : LB_\alpha^{ki}(x) \leq z, LB_q^{ki}(x) \leq z, z \in R, x \in T_i^k\right\} \tag{2}$$

   and $LB(T_i^k) = z^{ki}$
6. Set $M \leftarrow M \bigcup \{T_i^k : UB_k - LB(T_i^k) \geq \varepsilon, i = 1, 2\} \setminus \{T^k\}$

d) Updating step

- Update the lower bound: $LB_k = \min\{LB(T) : T \in M\}$.
- Delete from M all the intervals T such that $LB(T) > UB_k - \varepsilon$.

e) Stopping step

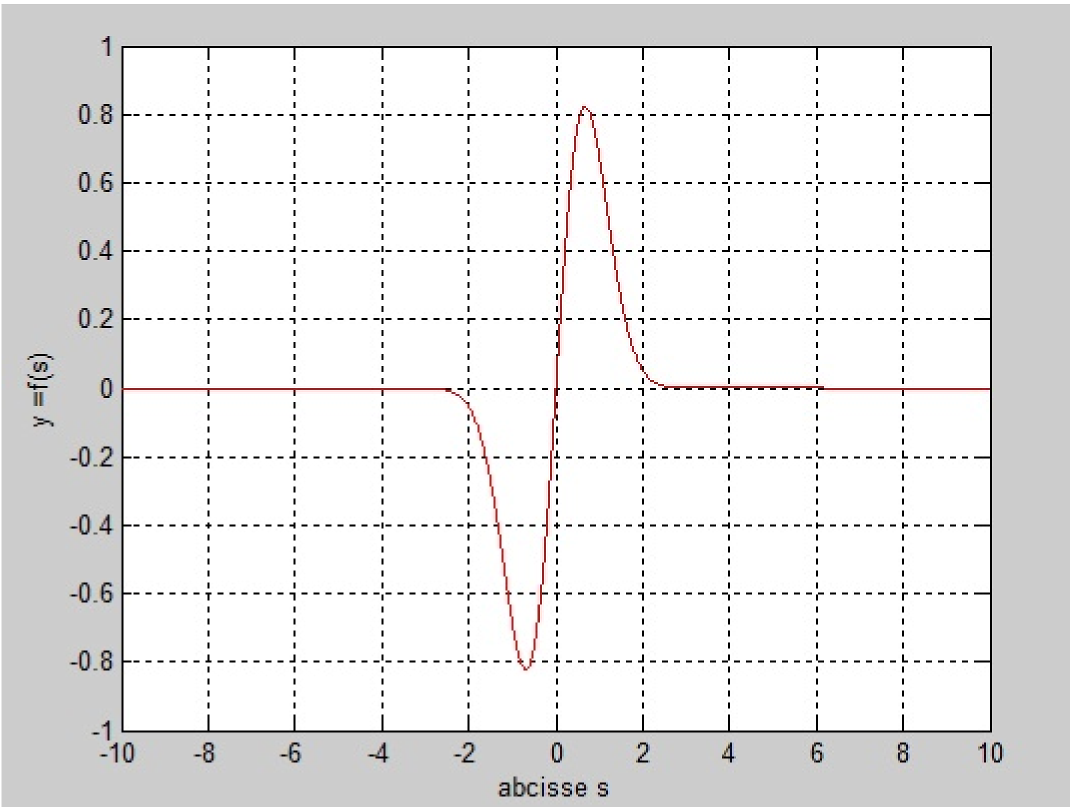
- **If** $M = \emptyset$ **then Output** $\bar{s}^k$ as an optimal solution and exit algorithm
- **else** set $k \leftarrow k + 1$, and return to **Step 2a)**.

Convergence of Algorithm

The purpose of the calculation artifices of types A_1 , A_2 and A_3 is to accelerate the convergence of the above Algorithm and the elements of demonstrations are as follows (see in https ://doi.org/10.1063/1.4992412, (2017).) .

Experimental Study

Test Problem f1 : b(s) = $b(s) = (s + sin(s)) * exp(-s^2)$ **,** $\forall s \in$ **[-10 , 10]**

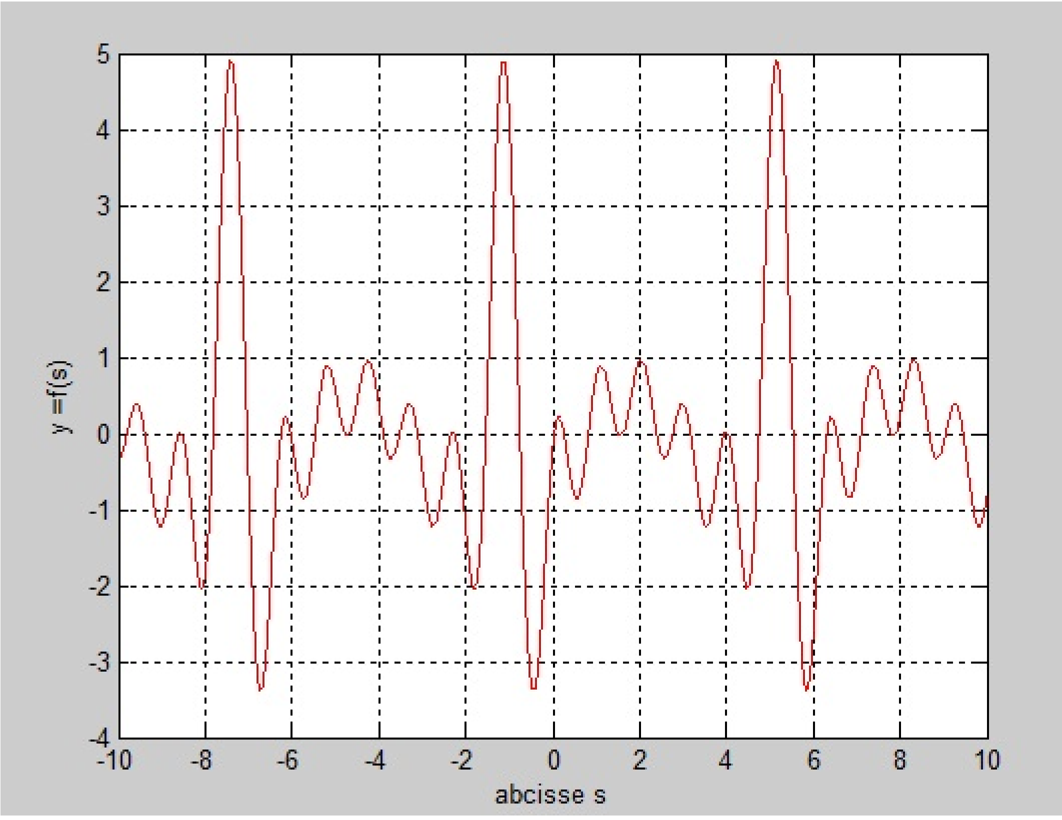


Progression of iterations ....

entiled of interval	Interval	Interval reduced	s^* (0ptimal)
T_0	[-10 , 10]	[-10 , 10]	-
T_{11}	[-10 0]	[-10 0]	-
T_{12}	[0 10]	[0 10]	-
T_{21}	[0 5]	[0 5]	-
T_{22}	[5 10]	[0 5]	-
T_{31}	[0 2.5241]	[-1.2224 -0.2612]	-
T_{32}	[2.5241 5]	[2.5241 5]	-
T_{41}	[-1.2224 -.74183]	[-0.789 -.7418]	-
T_{42}	[-0.74183 -0.2612]	[-0.7418 -0.2612]	-0.679576

Solution in 05 Iterations

Test Problem f2 : b(s)= $-sin((2)*s+1)-sin((3)*s+2)-sin((4)*s+3)-sin((5)*s+4)-sin((6)*s+5)$, $\forall s \in$ **[-10 , 10]**

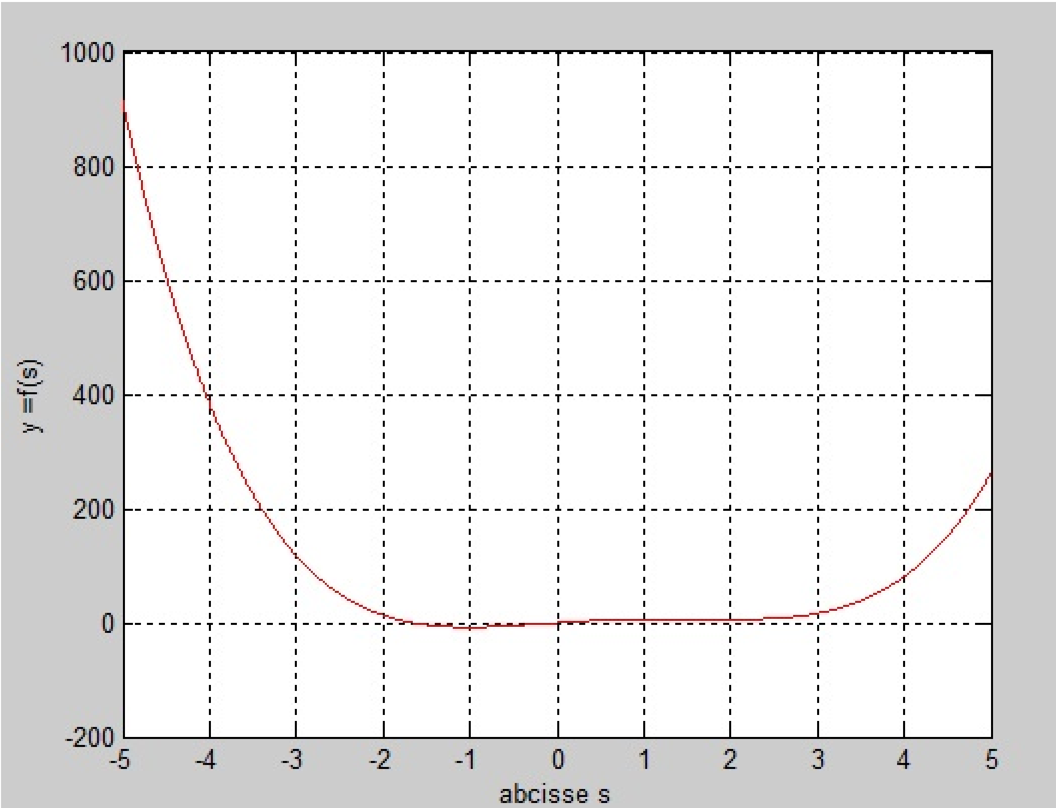


Progression of iterations ....

Iterations	Interval reduced	LB_k	UB_k	s^* (0ptimal)	Obs
1	[-9.2032 ,-8.8311]	-1.2168	-0.7511	-9.0276	Convexe
2	[-8.1341 , -8.1102]	-2.0114	-1.9602	-8.1102	Convexe
3	[-8.1102 , -7.8800]	-2.0354	-0.8251	-8.0804	Convexe
4	[-6.9879 , -6.6800]	-3.3729	-0.8253	-6.7201	Convexe
5	[-6.6800 , -6.6265]	-3.3179	-3.0890	-6.6799	Convexe
6	[-6.3721 , -6.0605]	-0.8244	0.1006	-6.3721	Concave
7	[-5.9017 , -5.6907]	-0.8454	-0.4738	-5.7290	Convexe
8	[-2.9200 , -2.5489]	-1.2168	-0.7550	-2.7444	Convexe
9	[-1.8509 , -1.8270]	-2.0114	-1.9601	-1.8270	Convexe
10	[-1.8270 , -1.5968]	-2.0354	-0.8251	-1.7972	Convexe
11	[-0.7047 , -0.3968]	-3.3729	-0.8253	-0.4369	Convexe
12	[-0.3968 , -0.3434]	-3.3178	-3.0891	-0.3967	Convexe
13	[0.5157 , 0.7332]	-0.8454	-0.4184	0.5542	Convexe
14	[4.3521 , 4.4708]	-2.0290	-1.6198	4.4708	Convexe
15	[4.4709 , 4.6235]	-2.0354	-1.4683	4.4860	Convexe
16	[5.5785 , 5.8864]	-3.3729	-0.8253	5.8463	Convexe
17	[5.8864 , 5.9398]	-3.3177	-3.0893	5.8865	Convexe
18	[6.1943 , 6.5058]	-0.8244	0.1009	6.1943	Concave
19	[6.6646 , 6.8756]	-0.8454	-0.4737	6.8374	Convexe
20	[9.5854 , 10.0000]	-1.2168	-0.5577	9.8220	Convexe

Solution in 20 Iterations

Test Problem f3 :b(s) $= b(s) = s^4 - 3 * s^3 - 1.5 * s^2 + 10 * s$ **,** $\forall s \in$ [**-5 , 5**]



Progression Of iterations ....

Entitled of Interval	Interval	Interval reduced	s^* (0ptimal)
T_0	[-5 , 5]	[-3.635 5]	-
T_{11}	[-3.635 0.6825]	[-1.8069 0.6825]	-
T_{12}	[0.6825 5]	[0.6825 1.6243]	-
T_{21}	[-1.8069 -0.5622]	[-1.8069 -0.5622]	-1.0000
T_{22}	[-0.5622 0.6825]	[-0.9157 -0.5622]	-

Solution in 03 Iterations

The second step: it is to solve a problem of any optimation in dimension $n > 1$ ***with separable variables***

We consider the following problem

$$(P)\begin{cases} \min f(x) = \sum_{i=1}^{n} f_i(x_i) \\ x \in D \subset R^n \end{cases}$$

* / The results above will help us to solve. among others
* / The techniques of parallelism
* / C.S.P Techniques
* / The techniques of artificial intelligence

The third step is to solve any optimization problem in dimension $n > 1$

We consider the following problem

$$(P)\begin{cases} \min f(x) \\ x \in D \subset R^n \end{cases}$$

* / The functions to be treated are of any types.
* / First of all, the functions of the holder.
* / To use any function, with any combination of functions (sin, cos, exp, log, power..........).

4 Conclusion

The study done in this paper proves a lot efficiency of our algorithm model. The experimental results prove the efficiency of our proposed method . The comparison of the results of our method was made compared to well-known methods in global optimization. The results were satisfactory .
In this paper we proposed a branch and prune algorithm for computing all global minimizers of univariate functions subject to bound constraints. The algorithm uses a combination of two lower bounds and utilizes a pruning technique as well as a convex/concave test in order to accelerate the search process. Numerical results show that the proposed method is efficient.

Our Algorithm supports a wide range of problems (mono - multi objectives) in global optimization, with signomial functions, sin, cos, arcsin, arcosin, arctang, sh, ch, ... powers, log, exp .... $\in R^n$.

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